Motion and Geometry in 3d
Parameterized curves in 3d

We can represent curves in three dimensional space with the parameterization

$$\mathbf{r}(\alpha) = (x(\alpha), y(\alpha), z(\alpha))$$

where $\alpha$ is a monotonically increasing parameter. Examples include distance along the curve, time traveled along the curve, or some other parameter.
Tangent and normal vectors

We can compute *tangent* and *normal* vectors using the following simple formulas:

\[
T(\alpha) = \frac{r'(\alpha)}{g(\alpha)}
\]

and

\[
N(\alpha) = \frac{g(\alpha)r''(\alpha) - g'(\alpha)r'(\alpha)}{g(\alpha)\kappa(\alpha)}
\]

where

\[
r'(\alpha) = (x'(\alpha), y'(\alpha), z'(\alpha))
\]

\[
r''(\alpha) = (x''(\alpha), y''(\alpha), z''(\alpha))
\]

and

\[
g(\alpha) = ||r'(\alpha)|| \quad \text{and} \quad \kappa(\alpha) = \frac{||r''(\alpha) \times r'(\alpha)||}{g(\alpha)^3}
\]
Tangent and normal vectors

We compute 2-norm \( \| \mathbf{r} \| \) for \( \mathbf{r} = (r_1, r_2, r_3) \) as

\[
\| \mathbf{r} \| = \sqrt{r_1^2 + r_2^2 + r_3^2}
\]

The cross product \( \mathbf{r} \times \mathbf{s} \) is computed as

\[
\mathbf{r} \times \mathbf{s} = (r_2s_3 - r_3s_2, -(r_1s_3 - r_3s_1), r_1s_2 - r_2s_1)
\]

The 2-norm is the length of vector \( \mathbf{r} \).

The cross product is perpendicular to both \( \mathbf{r} \) and \( \mathbf{s} \).

The 2-norm is a scalar whereas the cross product results in a vector.

In Matlab, we can use \( \text{norm}(\mathbf{r}, 2) \) and \( \text{cross}(\mathbf{r}, \mathbf{s}) \).
One additional operator that is useful is the *dot product* $\mathbf{r} \cdot \mathbf{s}$, defined as

$$\mathbf{r} \cdot \mathbf{s} = r_1 s_1 + r_2 s_2 + r_3 s_3$$

The 2-norm can be written in terms of the dot product as

$$\|\mathbf{r}\| = \sqrt{\mathbf{r} \cdot \mathbf{r}}$$

We also have

$$\mathbf{r} \cdot (\mathbf{r} \times \mathbf{s}) = \mathbf{s} \cdot (\mathbf{r} \times \mathbf{s}) = 0$$

In Matlab, we can use `dot(r,s)`. The cross product is perpendicular to both $\mathbf{r}$ and $\mathbf{s}$. 
Example

\[
x(\theta) = \cos(2\theta)(\cos(5\theta) + 3)
\]
\[
y(\theta) = \sin(2\theta)(\cos(5\theta) + 3)
\]
\[
z(\theta) = \sin(5\theta), \quad 0 \leq \theta \leq 2\pi
\]

We have

\[
r(\theta) = (x(\theta), y(\theta), z(\theta))
\]
\[
r'(\theta) = (x'(\theta), y'(\theta), z'(\theta))
\]

and

\[
r''(\theta) = (x''(\theta), y''(\theta), z''(\theta))
\]
A Trefoil knot
Trefoil knot with tangent and normals