1. For each of the following matrices, first find all eigenvalues and eigenvectors of the matrix. Then check that the product of the eigenvalues is equal to the determinant.

(a) \[
\begin{bmatrix}
0 & 2 \\
1 & 1
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
3 & 4 & 2 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0 \\
0 & 2 & 0 & 0 \\
2 & 0 & 0 & 0
\end{bmatrix}
\]

(d) The $3 \times 3$ matrix of all zeros

(e) The $4 \times 4$ identity matrix

2. Gibonacci numbers Consider the sequence $G_n$ which begins with $0, 1$ and then each successive entry is the average of the previous two. That is, $G_{n+1} = (G_n + G_{n-1})/2$.

(a) Write down a $2 \times 2$ matrix $A$ that performs this process.

(b) Diagonalize your matrix $A$

(c) Use your diagonalization to find a formula for $G_n$.

(d) Use your formula to compute the limit of the sequence $G_n$ as $n \to \infty$.

3. Suppose that each year, $25\%$ of Lineartown moves to Algebraville, and $15\%$ of Algebraville moves to Lineartown (and there are no other towns). What is the Markov matrix describing this behavior? What are its eigenvalues and eigenvectors? What is the eventual population distribution? (See Section 10.3 for how to use eigenvalue/eigenvector decompositions for studying population dynamics, economics and so on.)

4. Now suppose that there are three towns, with yearly population movements described below:
First write the Markov matrix for these movements. Second, show using calculations (or a proof) that \( \lambda = 1 \) is an eigenvalue of the Markov matrix. Third, find the eventual populations of the three towns. **Hint:** To show that \( \lambda = 1 \) is an eigenvalue, show that \( \det(A - I) = 0 \). You can do this because you know the columns of \( A \) sum to 1. (See Section 10.3 of your textbook).

5. For each of the following matrices \( A \), find the matrix \( e^A \).

   (a) \[ A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \]

   (b) \[ A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix} \]

6. Use your calculations from the previous question to solve the linear system of differential equations.

   (a) \( u' = u + v, \quad v' = 2v, \quad u(0) = u_0, \quad v(0) = v_0 \)

   (b) \( u' = 6u - 2v, \quad v' = 2u + v, \quad u(0) = v(0) = 30 \)

7. For each of the following matrices, diagonalize the matrix if possible. If you can’t then explain what went wrong.

   (a) \[ \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} \]

   (b) \[ \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \]

   (c) \[ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \]

   (d) \[ \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 13 \end{bmatrix} \]