Orthogonality and Projections

Math 301
Complementary spaces

We saw that there are four fundamental subspaces associated with the matrix $A \in \mathbb{R}^{m \times n}$.

- The column space and null space $C(A)$ and $N(A)$ of $A$.
- The column space and null space $C(A^T)$ and $N(A^T)$ of $A^T$.

**Question:** The column space of $A$ may not fill out all of $\mathbb{R}^m$. Can we find the “missing” vectors needed to form a basis for all of $\mathbb{R}^m$?

**Question:** The null space of $A$ may not fill out all of $\mathbb{R}^n$. Where are those missing basis vectors?
Complementary subspaces

For matrix $A \in \mathbb{R}^{m \times n}$, with rank $r$:

- The dimension of $C(A)$ is $\ldots \ r$
- The dimension of $N(A)$ is $\ldots \ n - r$
- The dimension of $C(A^T)$ is $\ldots \ r$
- The dimension of $N(A^T)$ is $\ldots \ m - r$

**Guess:**

To construct a basis for $\mathcal{R}^{m}$ we could use the $r$ basis vectors of $C(A)$ and the $m - r$ basis vectors of $N(A^T)$.
Complementary subspaces

For matrix $A \in \mathbb{R}^{m \times n}$, with rank $r$:

The dimension of $C(A)$ is \ldots $r$

The dimension of $N(A)$ is \ldots $n - r$

The dimension of $C(A^T)$ is \ldots $r$

The dimension of $N(A^T)$ is \ldots $m - r$

Guess:

To construct a basis for $\mathbb{R}^n$ we could use the $r$ basis vectors of $C(A^T)$ and the $n - r$ basis vectors of $N(A)$. 

True!
Two vectors $\mathbf{u}$ and $\mathbf{v}$ are called **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$ or equivalently, if $\mathbf{u}^T \mathbf{v} = 0$.

Two *subspaces* $\mathbf{U}$ and $\mathbf{V}$ are orthogonal if for every $\mathbf{u} \in \mathbf{U}$ and $\mathbf{v} \in \mathbf{V}$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, e.g. $\mathbf{u}^T \mathbf{v} = 0$.

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Example:

For a matrix $A$, the subspace $N(A)$ is orthogonal to $C(A^T)$.

Suppose $A\mathbf{x} = \mathbf{0}$. Then $\mathbf{x} \in N(A)$. Also, $\mathbf{x}$ is orthogonal to the rows of $A$, e.g. $\mathbf{x}$ is orthogonal to every vector in $C(A^T)$.
Example:

For a matrix $A$, the subspace $N(A^T)$ is orthogonal to $C(A)$.  
Suppose $A^T \mathbf{x} = 0$. Then $\mathbf{x} \in N(A^T)$. Also, $\mathbf{x}$ is orthogonal to the columns of $A$, e.g. $\mathbf{x}$ is orthogonal to every vector in $C(A)$. 

In both examples, the subspaces are *complements* of each other.

The subspace $N(A)$ contains *every* vector that is orthogonal to $C(A^T)$, and the subspace $N(A^T)$ contains every vector that is orthogonal to $C(A)$. 

Fundamental Theorem of Linear Algebra

- $N(A)$ is the orthogonal complement of the row space $C(A^T)$ (in $\mathbb{R}^n$).

- $N(A^T)$ is the orthogonal complement of the column space $C(A)$ (in $\mathbb{R}^m$).

Given a matrix $A \in \mathbb{R}^{m \times n}$, every vector $x$ in $\mathbb{R}^n$ can be written as the sum of something from $C(A^T)$ plus something from $N(A)$.

$$x = x_R + x_N, \quad x_R \in C(A^T), \quad x_N \in N(A)$$

The analogous claim holds for vectors from $\mathbb{R}^m$. 
In this light, the solvability of \( Ax = b \) can be rephrased as:

- Find an \( \hat{x} \) so that \( A\hat{x} \) in \( C(A) \) is ”closest” to \( b \).
- \textit{Project} the vector \( b \) onto \( C(A) \).
- Find only those components of \( b \) that are in \( C(A) \).
Decompose \( b \) into a “projection” piece \( p \) and an error piece \( e \). Note that the error piece \( e \) cannot be made any smaller.

\[
\mathbf{b} = \mathbf{p} + \mathbf{e}
\]

Find \( \mathbf{x} \) such that \( \mathbf{a} \cdot (\mathbf{b} - \mathbf{x} \mathbf{a}) = 0 \)

\[
\hat{x} = \frac{\mathbf{b}^T \mathbf{a}}{\mathbf{a}^T \mathbf{a}}, \quad \text{(scalar)}
\]

\( \mathbf{b} \) is not in the column space of \( \mathbf{a} \) and so we find the best approximation to \( \mathbf{b} \) in \( \mathbf{a} \).
Find $\hat{x}$ such that $A^T(b - A\hat{x}) = 0$

$$\hat{x} = (A^T A)^{-1} A^T b$$

The vector $p = A\hat{x}$ is in $C(A)$.

The vector $e = b - A\hat{x}$ is in $N(A^T)$.