Finding null space solutions

Math 301
Determine the solvability of a linear system, and characterize its solution.

We will be exclusively interested in systems with an infinite number of solutions.

Example: Solve

\[
\begin{bmatrix}
2 & 4 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 6
\]

Check:

\[
\begin{bmatrix}
2 & 4 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
3 \\
0
\end{bmatrix}
= 6
\]

We will learn how to find this solution

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
3 \\
0
\end{bmatrix} + \alpha \begin{bmatrix}
-2 \\
1
\end{bmatrix}
\]

“particular solution”

“Null space solution”
Characterizing infinite solutions

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} =
\begin{bmatrix}
  3 \\
  0
\end{bmatrix} + \alpha
\begin{bmatrix}
  -2 \\
  1
\end{bmatrix}
\]

This idea of a “point” (given by the particular solution) and a “direction” (given by the nullspace) generalizes to higher dimensions!

Note: This is also the graph of the line \(2x + 4y = 6\)!
Vector Spaces

Definition: The space $\mathbb{R}^n$ consists of all column vectors $\mathbf{v}$ with $n$ components.

Examples:

$\mathbb{R}^1: \mathbf{v} = [4], \mathbf{v} = [-1], \text{all other scalars (1 } \times \text{ 1 vectors).}$

$\mathbb{R}^2: \mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}, \text{ and in general } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\mathbb{R}^n: \mathbf{v} = [v_1, v_2, \ldots, v_n]^T, \text{ where } v_j \in \mathbb{R}.$
Vector subspaces

All lines through the origin form a vector subspace of $\mathbb{R}^n$

Why? If $\mathbf{v}$ and $\mathbf{w}$ are in the subspace, then all linear combinations $c\mathbf{v} + d\mathbf{w}$ are in the subspace.

Example: Consider the line $y = 5x$. What does the corresponding vector space look like?

The vector space consists of all multiples of $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Let $\mathbf{v} = \alpha_1[1, 5]^T$ and $\mathbf{w} = \alpha_2[1, 5]^T$.

Then $c\mathbf{v} + d\mathbf{w} = (c\alpha_1 + d\alpha_2)[1, 5]^T$ is a multiple of $[1, 5]^T$ and so is in the subspace.
Vector subspaces can be *generated* by a set of vectors. Let $S$ be a set of vectors, e.g.

$$S = \{(1, 4, 6), (-1, 3, 2), (3, 3, 1)\}$$

Then $S$ itself will not form a subspace (in general).

But all linear combinations of vectors in $S$ will form a subspace.

This generated subspace is the space *spanned* by the vectors in $S$. 
The columns of the matrix $A$ generate the *column space* $C(A)$.

**Key idea**: The linear system $Ax = b$ has a solution (”is solvable”) if and only if $b$ is in the column space of $A$.

Can $b$ be generated from the columns of $A$?

Only then will the system have a solution.
A second important subspace

The nullspace $N(A)$ of a matrix $A$ is the space of vectors $\mathbf{x}$ for which $A\mathbf{x} = 0$.

- Only singular matrices have a nullspace that contains more than just the zero vector.

Suppose that $\mathbf{v}$ and $\mathbf{w}$ are in the nullspace of $A$. Then show that all linear combinations of $\mathbf{v}$ and $\mathbf{w}$ are in the nullspace as well.

$$A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c(0) + d(0) = 0$$

The zero vector is clearly in the nullspace, and so the nullspace is a subspace.
Example 1

Find the \textit{nullspace} of \(A = \begin{bmatrix} 2 & 4 \end{bmatrix}\).

We solve the linear system

\[
\begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0
\]

Write the system as \(2x = -4y\).
Solve \(2x = -4y\) by setting \(y = 1\) and getting \(x = -2\).

The \textit{nullspace} of \(A = \begin{bmatrix} 2 & 4 \end{bmatrix}\) are the multiples of the special solution

\[
s_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]
Find the solutions to the system:

\[
\begin{bmatrix}
1 & 3 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = 0
\]

Write the system as \( x = -3y - 5z \).

The variable \( x \) is the ”pivot” variable and \( y \) and \( z \) are the ”free” variables.

Choose \( y \) and \( z \) so that \( x \) is easy to evaluate.

\( s_1 = \begin{bmatrix}
-3 \\
1 \\
0
\end{bmatrix} \), \( s_2 = \begin{bmatrix}
-5 \\
0 \\
1
\end{bmatrix} \)
The nullspace of $A = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$ are all linear combinations of the special solutions

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

Show that any linear combination of $s_1$ and $s_2$ are in the nullspace:

$$\mathbf{x} = cs_1 + ds_2 \quad \rightarrow \quad A\mathbf{x} = 0$$

The zero vector is in the nullspace.

$$\mathbf{x} = 0 \quad \rightarrow \quad A\mathbf{x} = 0$$
Example 3

Find the nullspace of the matrix

\[
A = \begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
1 & 2 & 3 & 6 & 9 \\
0 & 0 & 1 & 2 & 3
\end{bmatrix}
\]

Use row reduction to reduce the matrix to an “echelon matrix”

\[
U = \begin{bmatrix}
1 & 2 & 2 & 4 & 6 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The ”pivot” variables are \(x_1\) and \(x_3\). The free variables are \(x_2\), \(x_4\) and \(x_5\).

We can now solve an equivalent system \(U\mathbf{x} = 0\).
Example 3 - continued

\[
U = \begin{bmatrix}
1 & 2 & 2 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
4 & 6 \\
2 & 3 \\
0 & 0
\end{bmatrix}
\]

Write the system as

\[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_3
\end{bmatrix} = - \begin{bmatrix}
2 & 4 & 6 \\
0 & 2 & 3
\end{bmatrix} \begin{bmatrix}
x_2 \\
x_4 \\
x_5
\end{bmatrix}
\]

Pivot variables

Free variables

We can ignore the last row since it contains all zeros

Set free variables to columns of the 3x3 identity matrix and solve for the pivot variables to get 3 special solutions.
To get $s_1$, we set $x_2 = 1$, $x_4 = x_5 = 0$, and solve for $x_1$ and $x_3$:

\[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_3
\end{bmatrix} = -\begin{bmatrix}
2 \\
0
\end{bmatrix}
\]

We get $x_1 = -2$ and $x_3 = 0$. So the special solution is

\[
s_1 = \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}
\]
Example 3 - continued

To get $s_2$, we set $x_4 = 1$, $x_2 = x_5 = 0$, and solve for $x_1$ and $x_3$:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = -\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

We get $x_1 = 0$ and $x_3 = -2$. So the special solution is

$$s_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$
To get $s_3$, we set $x_5 = 1$, $x_2 = x_4 = 0$, and solve for $x_1$ and $x_3$:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = -\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

We get $x_1 = 0$ and $x_3 = -3$. So the special solution is

$$s_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$
Example 3 - continued

The nullspace of the matrix

\[
A = \begin{bmatrix}
    1 & 2 & 2 & 4 & 6 \\
    1 & 2 & 3 & 6 & 9 \\
    0 & 0 & 1 & 2 & 3
\end{bmatrix}
\]

consists of all linear combinations of the special solutions

\[
s_1 = \begin{bmatrix}
    -2 \\
    1 \\
    0 \\
    0 \\
    0
\end{bmatrix}, \quad s_2 = \begin{bmatrix}
    0 \\
    0 \\
    -2 \\
    1 \\
    0
\end{bmatrix}, \quad s_3 = \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    -3 \\
    1
\end{bmatrix}
\]

Free variables are chosen to be columns of the identity matrix (for convenience only!)

We must solve for the pivot variables.
Example 3 - continued

We can take U all the way to “row-reduced echelon form” by eliminating non-zero entries above the pivot entries.

\[ U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

reduces to

\[ R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

and we can write our system as

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = -\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix}
\]