Numerical Linear Algebra – Homework 3

This assignment is due at 3:15 p.m. on Thursday 9 October.

1) Let \( \| \cdot \| \) be a vector norm. Prove:
\[
\| \|x\| - \|y\| \| \leq \| x - y \|
\]

2) Let \( x \) be a vector in \( \mathbb{R}^n \). Prove that the vector 1-norm
\[
\|x\|_1 = \sum_{i=1}^{n} |x_i|
\]
is indeed a vector norm.

3) Let \( \| \cdot \| \) be a matrix \( p \)-norm. Let \( A \) and \( B \) both be \( n \times n \) matrices. Prove:
\[
\|AB\| \leq \|A\| \|B\|
\]

4) Let \( A \) be an \( n \times n \) matrix. Let \( \lambda \) be an eigenvalue of \( A \) with associated eigenvector \( x \). Let \( \alpha \) and \( \beta \) be scalars. Prove that \( \alpha \lambda + \beta \) is an eigenvalue of the matrix \( \alpha A + \beta I \) with associated eigenvector \( x \).

5) **Required for only those students enrolled in MATH 597.** In class, the matrix \( p \)-norm was defined as
\[
\|A\|_p = \max_{\|x\| \neq 0} \frac{\|Ax\|_p}{\|x\|_p}
\]
Show that
\[
\|A\|_p = \max_{\|x\|=1} \|Ax\|_p
\]
is an appropriate alternative definition.