Numerical Linear Algebra – Final Examination

This exam is due at 5:30 p.m. on Tuesday 16 December.

1) Let A be a real $n \times n$ symmetric positive definite matrix. Define $\varphi : R^n \to R$ by

$$\varphi(\mathbf{x}) = \sqrt{\mathbf{x}^T A \mathbf{x}}.$$

Prove that φ is a vector norm. (Hints: Factor A and use the Cauchy-Schwarz inequality.)

2) Let B be a *skew-Hermitian* matrix (i.e., $B^* = -B$).

a) Give a non-trivial example of a complex 2×2 skew-Hermitian matrix.

b) Show that all eigenvalues of an $n \times n$ skew-Hermitian matrix lie on the imaginary axis of the complex plane.

c) Let S be an $n \times n$ skew-Hermitian matrix. Show that $I \pm \alpha S$ is non-singular, where α is a real non-zero scalar.

d) Let S be an $n \times n$ skew-Hermitian matrix. Show that $Q = (I+S)(I-S)^{-1}$ is a unitary matrix.

3) Let A be a real 2×2 symmetric positive definite matrix. Consider the use of the Jacobi method to solve $A\mathbf{x} = \mathbf{b}$. Must the Jacobi method must converge? If yes, prove it. If not, provide a counterexample.

4) **Required only for those enrolled in MATH 597.** Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ and (respectively) associated eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)}, \ldots, \mathbf{v}^{(n)}$. Let $\mathbf{x}$ be a vector such that $\mathbf{x}^T \mathbf{v}^{(1)} = 1$. Let

$$B = A - \lambda_1 \mathbf{v}^{(1)} \mathbf{x}^T.$$

Prove that the eigenvalues of B are $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ with (respectively) associated eigenvectors $\mathbf{v}^{(1)}, \mathbf{w}^{(2)}, \mathbf{w}^{(3)}, \ldots, \mathbf{w}^{(n)}$, where the vectors $\mathbf{v}^{(i)}$ and $\mathbf{w}^{(i)}$ are related via the equation

$$\mathbf{v}^{(i)} = (\lambda_i - \lambda_1)\mathbf{w}^{(i)} + \lambda_1 \left(\mathbf{x}^T \mathbf{w}^{(i)}\right) \mathbf{v}^{(1)},$$

$i = 2, 3, \ldots\ n$.

5) Let A be an $n \times n$ matrix. Let λ be a non-zero eigenvalue of A with associated eigenvector $\mathbf{v}$. Define

$$\mathbf{x} = \frac{1}{\lambda v_i^{(1)}} \begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,n} \end{bmatrix},$$

where $v_i^{(1)} \neq 0$. Prove $\mathbf{x}^T \mathbf{v}^{(1)} = 1$.

6) Write Matlab code to find all the eigenvalues and, for each eigenvalue, an associated eigenvector of a real $n \times n$ matrix. You may assume all the eigenvalues are real and distinct. Follow the following procedure.

1. Do a few iterations of the power method to obtain rough approximations for the dominant eigenvalue and an associated eigenvector.
2. Switch to the inverse power method to get better approximations for the dominant eigenvalue and an associated eigenvector. Run the inverse power method to convergence.
3. Use the deflation technique discussed in class so that you may repeat steps 1 and 2 to obtain the next most dominant eigenvalue and associated eigenvector.
4. Repeat until you have approximations for all eigenvalues and, for each, an associated eigenvector.
5. To polish them, run your eigenvalues and eigenvectors through the inverse power method with the original matrix.
6. **Be liberal with comments in your code. I should not have to work to follow your logic!**