MATH 187 - Test 3 - Key

1) Prove that consecutive integers must be rel prime.

pf Let $x, x+1$ be consecutive integers. We are done when we show that $\exists\ a, b \in \mathbb{Z}$ s.t. $ax + b(x+1) = 1$

Let $a = -1$, $b = 1$.

Then we get $-1(x) + 1(x+1) = -x + x + 1 = 1$.

So we are done. $\boxtimes$

2) a) Find $\gcd(32, 81)$

$81 = 2 \cdot 32 + 17$
$32 = 1 \cdot 17 + 15$
$17 = 1 \cdot 15 + 2$
$15 = 7 \cdot 2 + 1$
$2 = 2 \cdot 1 + 0$ so $\gcd(32, 81) = 1$

b) Find $x, y \in \mathbb{Z}$ s.t $32x + 81y = 1$

$$\begin{aligned}
1 &= 15 - 7 \cdot 2 \\
&= 15 - 7(17 - 15) \\
&= -7 \cdot 17 + 8 \cdot 15 \\
&= -7 \cdot 17 + 8(32 - 17) \\
&= 8 \cdot 32 - 15 \cdot 17 \\
&= 8 \cdot 32 - 15(81 - 2 \cdot 32) \\
&= -15 \cdot 81 + 38 \cdot 32 \quad ☺
\end{aligned}$$

so $x = 38$, $y = -15$

c) In $\mathbb{Z}_{81}$, ☺ reduces to $1 = 38 \otimes 32$

so $32^{-1} = 38$

so $3 \oslash 32 = 3 \otimes 32^{-1}$
$= 3 \otimes 38$
$= 114 \bmod 81$
$= 33$

3) $\begin{cases} x \equiv 4 \mod 5 \\ x \equiv 7 \mod 11 \end{cases}$

$x \equiv 4 \pmod 5 \Rightarrow x = 5k+4$ for $k \in \mathbb{Z}$

so $5k+4 \equiv 7 \pmod{11}$

$5k \equiv 3 \pmod{11}$

so $5 \otimes k = 3$ in $\mathbb{Z}_{11}$. what is 5^{-1} in $\mathbb{Z}_{11}$?

note $5 \cdot (-2) = -10 \equiv 1 \pmod{11}$

so $5 \cdot 9 = 45 \equiv 1 \pmod{11}$

so $5^{-1} = 9$ in $\mathbb{Z}_{11}$

so $9 \otimes 5 \otimes k = 9 \otimes 3$

$k = 27 \mod 11$

$k = 5$

so $x = 25 + 4$

$x = 29$

But this is not the only solution.

All the solutions are of the form

$x = 29 + 55i, \quad i \in \mathbb{Z}.$

4) Let $a, b, c \in \mathbb{Z}^+$. Let $a | bc$ and suppose a and b are rel prime.

Prove $a | c$.

pf Since $a | bc$, $\exists k \in \mathbb{Z}$ s.t. $ak = bc$

Since a and b are rel prime, $\exists x, y \in \mathbb{Z}$ s.t. $ax + by = 1$

so $acx + bcy = c$

$acx + aky = c$

$a(cx + ky) = c$. Since $cx + ky \in \mathbb{Z}$, this says that

$a | c$. $\boxtimes$