ANALYTICAL UPSTREAM COLLOCATION SOLUTION OF A FORCED STEADY-STATE CONVECTION-DIFFUSION EQUATION

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Abstract

We give herein formulas for the solution of the Hermite collocation discretization of a nonhomogeneous steady-state convection-diffusion equation in one spatial dimension and with constant coefficients, defined on a uniform mesh, with Dirichlet boundary conditions. The accuracy of the method is enhanced by employing "upstream weighting" of the convective term in an optimal way. We discuss also the issue of where to optimally sample the forcing function. Computational examples illustrate the efficacy of the optimal upstream weighting technique combined with optimal sampling of the forcing function.

1 Introduction

It is well known that the numerical solution of convection-diffusion differential equations (DEs) can be a difficult task when convection is the dominant process. Numerical techniques often give rise to spurious oscillations that are not present in the continuous (i.e., not numerical/discrete) solution of the DE. To ameliorate these often physically unmeaningful (and therefore undesirable) oscillations, the technique of upstream weighting is often used ([1], [3], [4], [5], [7]). While upstreaming can eliminate the oscillations, it is often at the expense of "smearing" the sharp solution profile of the continuous solution of the DE.

In a previous work [3], we studied the homogeneous convection-diffusion equation

$$-D\frac{d^2u}{dx^2} + v\frac{du}{dx} = 0 \tag{1}$$

with Dirichlet boundary conditions, defined on a uniform mesh on the interval $[0,1]$, discretized via Hermite collocation. The convection coefficient v and diffusion coefficient D are both positive constants. We employed upstream weighting in the evaluation of the convective term in an optimal way, obtaining excellent agreement between the exact and numerical solutions, especially for large Péclet numbers.

In this paper, we extend our results to the nonhomogeneous equation

$$-D\frac{d^2u}{dx^2} + v\frac{du}{dx} = S(x). \tag{2}$$

In particular, we give formulas for the collocation solution of (2) defined on a uniform mesh with Dirichlet boundary conditions. We employ the optimal upstream weighting derived and studied in [3] and discuss where to optimally sample the forcing function $S(x)$ in (2). For the case where $S(x)$ is a linear function, we prove that the optimal sampling locations coincide with the optimal upstream locations.

Collocation discretization of the transient convection-diffusion equation has been studied in a variety of papers. Allen [1], using Taylor series analysis, explains why Hermite collocation can eliminate the unwanted oscillations provided upstream weighting is utilized but does not address simultaneously eliminating the "smearing" effect. Pinder and Shapiro ([5], [7]) consider replacing the traditional cubic Hermite basis with quartic polynomials, successfully eliminating the oscillations but not the residual smearing. They also provide a Fourier analysis for their method. And, in previous papers [2] and [3], the present author derived analytical formulas for the Hermite collocation solution of (1).

This paper is organized as follows. We first describe Hermite collocation in the context of the DE (2), utilizing optimal upstream weighting. We then provide formulas for the solution of the matrix equation that arises from the discretization. We compare these formulas for the numerical solution of (2) with the corresponding formulas for the exact solution of (2), both in general and when the forcing function $S(x)$ is linear. We then provide computational examples which illustrate the theory presented herein. A short section summarizing our results concludes the paper.

2 Hermite Collocation with Optimal Upstreaming

The differential equation (2) is defined on the interval $[0,1]$ with Dirichlet boundary conditions included. We partition the interval $[0,1]$ into m uniform subintervals $[0=x_0,x_1], [x_1,x_2], \ldots, [x_{m-1},x_m=1]$. Let $h=\frac{1}{m}=x_{j+1}-x_j$, $j=0,1,\ldots,m-1$. The Péclet number β, which plays a fundamental role in what follows, is defined

$$\beta = \frac{hv}{D} = \frac{v}{mD.}$$

The discretization proceeds by introducing a piecewise cubic Hermite interpolating polynomial

$$\widehat{u}(x) = \sum_{j=0}^{m}\left[u_j f_j(x) + u_j' g_j(x)\right] \tag{3}$$

into the DE (2), obtaining

$$-D\frac{d^2\widehat{u}}{dx^2} + v\frac{d\widehat{u}}{dx} - S(x) = E(x), \tag{4}$$

where $E(x)$ is an error function.

The Hermite basis functions, defined for $\eta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, are

$$f_j(x) = \begin{cases} \frac{1}{2}(1+2\eta)^2(1-\eta), & x_{j-1} \le x = x_j + \left(\eta - \frac{1}{2}\right)h \le x_j \\ \frac{1}{2}(1-2\eta)^2(1+\eta), & x_j \le x = x_j + \left(\eta + \frac{1}{2}\right)h \le x_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

and

$$g_j(x) = \begin{cases} \frac{h}{8}(2\eta+1)^2(2\eta-1), & x_{j-1} \le x = x_j + \left(\eta - \frac{1}{2}\right)h \le x_j \\ \frac{h}{8}(2\eta-1)^2(2\eta+1), & x_j \le x = x_j + \left(\eta + \frac{1}{2}\right)h \le x_{j+1} \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Note that $\widehat{u}$ in (3) interpolates the values $u_j = u(x_j)$ and $u'_j = \frac{du}{dx}(x_j)$, $j = 0, 1, \ldots, m$, because $f_j(x_k) = \delta_{jk}$, $\frac{df_j}{dx}(x_k) = 0$, $g_j(x_k) = 0$, and $\frac{dg_j}{dx}(x_k) = \delta_{jk}$. Here δ_{jk} is the Kronecker symbol.

It is clear that (4) has $2(m+1)$ coefficients, namely u_j and u'_j, for $j = 0, 1, 2, \ldots m$. However, the imposition of boundary conditions reduces this number to $2m$. To generate the $2m$ equations necessary to find these undetermined coefficients, the traditional choice is to enforce that the error function $E(x)$ in (4) is identically zero at two distinct "collocation points" in the interior of each of the m subintervals.

Given certain smoothness conditions, the optimal (in terms of minimizing local discretization error) location of the collocation points within each subinterval corresponds to the points of Gaussian quadrature [6], which in turn corresponds to choosing the collocation points as $\eta = \pm\frac{1}{\sqrt{12}}$ (see (5) and (6)) in each subinterval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (given in local η coordinates). Collocating at these Gauss points is often called "orthogonal" collocation. As shown in [6], this provides $\mathcal{O}(h^4)$ local discretization error. In this work and in [1] and [3], this choice corresponds to an absence of upstream weighting. However, the presence of a large Péclet number violates the smoothness conditions stipulated in [6]; thus the Gauss points are not, in general, optimal for our DE. As seen in [3] and reiterated below, use of the Gauss points (i.e. no upstream weighting) is optimal only for a very small range of values of β.

Upstream weighting is implemented in the following manner, which was introduced in [1] and utilized in [3]. As we mentioned above, we have $2m$ equations in $2m$ unknowns, where the $2m$ equations are traditionally generated by forcing $E(x) = 0$ in (4) at two collocation points in each of the m subintervals $[x_j, x_{j+1}]$, $j = 0, 1, 2, \ldots, m-1$. When implementing upstreaming, we still enforce $E(x) = 0$ for each of our $2m$ equations and we still evaluate $\frac{d^2\widehat{u}}{dx^2}$ in (4) at the Gauss points

$\eta = \pm\frac{1}{\sqrt{12}}$. However, we evaluate $\frac{d\widehat{u}}{dx}$ at the points $\eta = \pm\frac{1}{\sqrt{12}} - \zeta$, where $\zeta \geq 0$ controls how much upstreaming occurs. Because the support of each basis function f_j or g_j (see (5) and (6)) is the interval $[-\frac{1}{2}, \frac{1}{2}]$, it is clear that ζ must lie in the interval $\left[0, \frac{1}{2} - \frac{1}{\sqrt{12}}\right]$.

It should be noted that the discussion above gives no guidance about where to evaluate the forcing function $S(x)$. We will study this issue, for the case where $S(x)$ is a linear function, in Section 4.

It is straightforward to see that choosing the collocation points in this manner leads to a matrix equation with the computational molecule

$$\begin{bmatrix} M_{11} & M_{12} & -M_{11} & M_{14} \\ M_{21} & M_{22} & -M_{21} & M_{24} \end{bmatrix} \begin{bmatrix} q_j \\ r_j \\ q_{j+1} \\ r_{j+1} \end{bmatrix} = \begin{bmatrix} S_{2j} \\ S_{2j+1} \end{bmatrix}, \tag{7}$$

$j = 0, 1, 2, \ldots, m-1$. Here $q_j = u_j$, $r_j = u'_j$ (see (3)), and $S_k = S(x_k)$, $k = 0, 1, \ldots, 2m-1$, where these x_k's are the locations at which the forcing function $S(x)$ is evaluated. Note that the matrix equation represented by (7) is a system of $2m$ equations in $2(m+1)$ unknowns. The entries of the matrix are

$$\begin{aligned}
M_{11} &= \frac{2\sqrt{3}D}{h^2} + \frac{v}{h}\left(6\zeta^2 + 2\sqrt{3}\zeta - 1\right) \\
M_{21} &= -\frac{2\sqrt{3}D}{h^2} + \frac{v}{h}\left(6\zeta^2 - 2\sqrt{3}\zeta - 1\right) \\
M_{12} &= \frac{D}{h}(1+\sqrt{3}) + v\left(\frac{\sqrt{3}}{6} + \zeta + \sqrt{3}\zeta + 3\zeta^2\right) \\
M_{22} &= \frac{D}{h}(1-\sqrt{3}) + v\left(-\frac{\sqrt{3}}{6} + \zeta - \sqrt{3}\zeta + 3\zeta^2\right) \\
M_{14} &= \frac{D}{h}(-1+\sqrt{3}) + v\left(-\frac{\sqrt{3}}{6} - \zeta + \sqrt{3}\zeta + 3\zeta^2\right) \\
M_{24} &= -\frac{D}{h}(1+\sqrt{3}) + v\left(\frac{\sqrt{3}}{6} - \zeta - \sqrt{3}\zeta + 3\zeta^2\right)
\end{aligned}$$

which reduce to those given in [2] for the case of $\zeta = 0$.

For the case where the forcing function $S(x)$ is identically zero, we derived in [3] an algorithm which tells us how to choose ζ optimally as a function of β. This algorithm and its depiction appear in Table 1 and Figure 1, respectively. (In Table 1, ϵ is a small positive number. For the numerical work in [3] and herein, its value is set to 10^{-6}.) As we can see, the use of the Gauss points for collocation (i.e., when $\zeta = 0$) is optimal only for $\beta \in \left(0, 2\sqrt{3}\right]$.

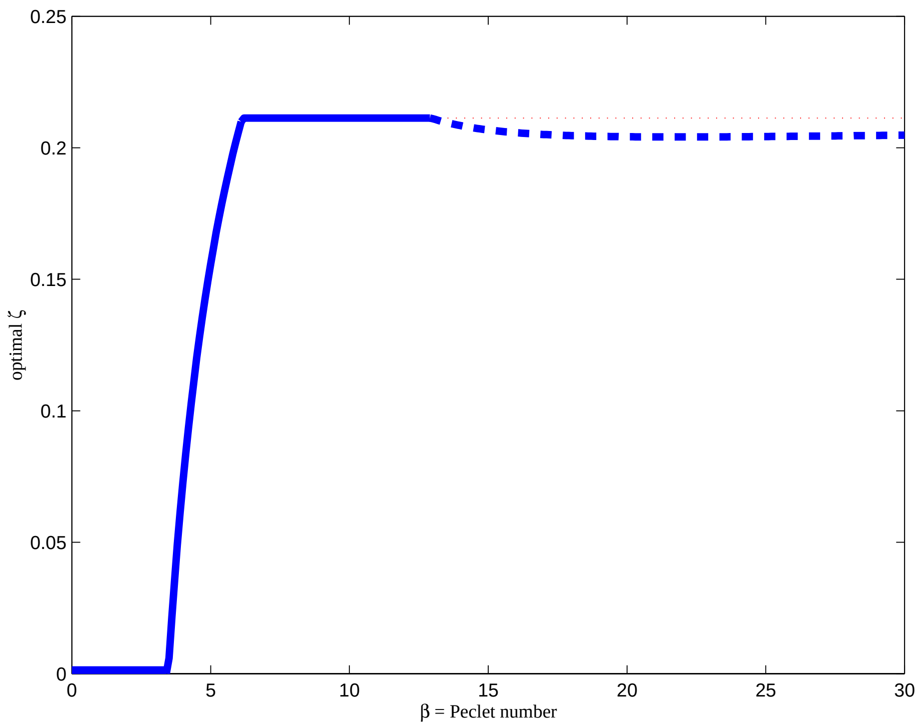

Figure 1: Optimal ζ as a function of Péclet number β

Table 1: Optimal ζ as a function of β.

β interval	approx β interval	optimal ζ
$(0, 2\sqrt{3}]$	$(0, 3.46410]$	0
$[2\sqrt{3}, \sqrt{3} + 2^{-1/2}(3^{3/4} + 3^{5/4})]$	$[3.46410, 6.13572]$	$\frac{\sqrt{6\beta^2 - 36} - 6}{6\beta}$
$[\sqrt{3} + 2^{-1/2}(3^{3/4} + 3^{5/4}), 6 + 4\sqrt{3}]$	$[6.13572, 12.9282]$	$\frac{1}{2} - \frac{1}{\sqrt{12}}$
$[6 + 4\sqrt{3}, \infty)$	$[12.9282, \infty)$	$\frac{1}{2} - \frac{2}{\beta} - \frac{\sqrt{\beta^2 - 12\beta + 24}}{\sqrt{12}\beta} - \epsilon$

3 Analytical Solution of Forced Upstream Collocation

We now state the main result of this paper. Its proof is straightforward, but computationally tedious.

Theorem 3.1 *The general solution of (7), which represents the upstream collocation discretization of (2) with Dirichlet boundary conditions and with a uniform mesh of m subintervals imposed on the interval $[0, 1]$, is given by*

$$q_j = c_1 + c_2\,\lambda^j + \frac{1}{2m}\sum_{k=0}^{2m-1} G_{k,j} S_k \tag{8}$$

$$r_j = \rho\, c_2\,\lambda^j + \frac{1}{2m}\sum_{k=0}^{2m-1} G'_{k,j} S_k, \tag{9}$$

where c_1 and c_2 are constants determined from boundary conditions,

$$\lambda = \frac{\beta^2 + 6\beta + 12 + 6\beta\zeta(4 + \beta + \beta\zeta)}{\beta^2 - 6\beta + 12 + 6\beta\zeta(4 - \beta + \beta\zeta)}, \tag{10}$$

$$\rho = \frac{2\beta m(1 + \beta\zeta)}{\beta^2\zeta^2 + 4\beta\zeta + 2}, \tag{11}$$

and S_k is as in (7). The constants c_1 and c_2 are given by

$$c_2 = \frac{u(1) - u(0)}{\lambda^m - 1} \tag{12}$$

and

$$c_1 = u(0) - c_2.$$

The discrete Green's functions $G_{k,j}$ and $G'_{k,j}$ are

$$G_{k,j} = \begin{cases} A_k(\lambda^m - \lambda^j), & k = 0, 1, \ldots, 2j-1 \\ (C_k - A_k\lambda^m)(\lambda^j - 1), & k = 2j, 2j+1, \ldots, 2m-1 \end{cases} \tag{13}$$

and

$$G'_{k,j} = \begin{cases} -\rho A_k \lambda^j, & k = 0, 1, \ldots, 2j-1 \\ \rho(C_k - A_k\lambda^m)\lambda^j, & k = 2j, 2j+1, \ldots, 2m-1, \end{cases} \tag{14}$$

where

$$A_{2\ell} = \frac{(1 + \beta\zeta - \sqrt{3}\beta\zeta^2)\lambda_{num}\lambda^\ell + \rho_{den}(-6 - \sqrt{3}\beta - 6\beta\zeta + 6\sqrt{3}\beta\zeta^2)}{v(1+\beta\zeta)\lambda_{num}\lambda^\ell(\lambda^m - 1),} \tag{15}$$

$$A_{2\ell+1} = \frac{(1 + \beta\zeta + \sqrt{3}\beta\zeta^2)\lambda_{num}\lambda^\ell + \rho_{den}(-6 + \sqrt{3}\beta - 6\beta\zeta - 6\sqrt{3}\beta\zeta^2)}{v(1+\beta\zeta)\lambda_{num}\lambda^\ell(\lambda^m - 1),} \tag{16}$$

and

$$C_k = \begin{cases} \dfrac{1 + \beta\zeta - \sqrt{3}\beta\zeta^2}{v(1+\beta\zeta)}, & k \ even \\ \\ \dfrac{1 + \beta\zeta + \sqrt{3}\beta\zeta^2}{v(1+\beta\zeta)}, & k \ odd. \end{cases} \tag{17}$$

The symbols λ_{num} and ρ_{den} refer respectively to the numerator of (10) and denominator of (11).

It is instructive to observe the relationships between the formulas given in Theorem 3.1 and the exact solution to (2) with Dirichlet boundary conditions on the interval $[0, 1]$:

$$u(x) = k_1 + k_2\, e^{\beta m x} + \int_0^1 G(\xi, x)\, S(\xi)\, d\xi \tag{18}$$

where

$$k_2 = \frac{u(1) - u(0)}{e^{\beta m} - 1,} \tag{19}$$

$$k_1 = u(0) - k_2,$$

and the (continuous) Green's function is

$$G(\xi, x) = \begin{cases} \dfrac{\left[e^{\beta m \xi} - 1\right]\left[e^{\beta m(1-\xi)} - e^{\beta m(x-\xi)}\right]}{v(e^{\beta m} - 1)}, & 0 \le \xi \le x \\ \\ \dfrac{\left[e^{\beta m x} - 1\right]\left[e^{\beta m(1-\xi)} - 1\right]}{v(e^{\beta m} - 1)}, & x \le \xi \le 1. \end{cases}$$

If we evaluate (18) and its derivative at the jth mesh point $x_j = \frac{j}{m}$, we obtain

$$u(x_j) = k_1 + k_2 e^{\beta j} + \int_0^1 G(\xi, x_j)\, S(\xi)\, d\xi \tag{20}$$

where

$$G(\xi, x_j) = \begin{cases} A(\xi)\left[e^{\beta m} - e^{\beta j}\right], & 0 \le \xi \le x_j \\ \\ \left[\dfrac{1}{v} - A(\xi)e^{\beta m}\right](e^{\beta j} - 1), & x_j \le \xi \le 1 \end{cases} \tag{21}$$

and

$$\frac{du}{dx}(x_j) = \beta m\, k_2\, e^{\beta j} + \int_0^1 \frac{\partial G}{\partial x}(\xi, x_j)\; S(\xi)\, d\xi, \tag{22}$$

where

$$\frac{\partial G}{\partial x}(\xi, x_j) = \begin{cases} -A(\xi)\,\beta m\, e^{\beta j}, & 0 \le \xi < x_j \\ \\ \left[\dfrac{1}{v} - A(\xi)\, e^{\beta m}\right] \beta m\, e^{\beta j}, & x_j < \xi \le 1. \end{cases} \tag{23}$$

(Formula (22) is obtained using the Leibniz rule for differentiation of integrals and using the fact that (21) is continuous at $\xi = x_j$.) The function $A(\xi)$ is

$$A(\xi) = \frac{1 - e^{-\beta m \xi}}{v(e^{\beta m} - 1).} \tag{24}$$

To see the relationships alluded to above, we begin by mentioning that λ (see (10)) is the discrete analogue of e^{β}. (In fact, this point was studied in detail in [2], where we showed that the McLaurin series expansions of λ (when $\zeta = 0$) and e^{β} agreed up to and including the β^4 term, an observation which meshes perfectly with a fact mentioned above in Section 2, namely that orthogonal (i.e., $\zeta = 0$) collocation provides $\mathcal{O}(h^4)$ local discretization error.) This correspondence between λ and e^{β} is evident in many places, including comparing the arbitrary constants c_2 (see (12)) and k_2 (see (19)) and comparing the discrete solution q_j (see (8)) and exact solution $u(x_j)$ (see (20)). Further comparison of (8) and (20) reveals that the role of the definite integral in (20) is assumed by the numerical integration rule defined by the summation in (8). It is also evident that the discrete Green's functions (13) and (14) correspond to their counterparts (21) and (23). Continuing with these last comparisons, ρ (see (11)) is the analogue of βm (indeed $\rho = \beta m$ when $\zeta = 0$) and C_k (see (17)) is the analogue of $\frac{1}{v}$ (and, indeed $C_k = \frac{1}{v}$ when $\zeta = 0$). Finally, (15) and (16) clearly correspond to (24), but this relationship is more tenuous than those discussed above due to the fact that ξ is a "dummy" variable and thus has no direct connection to the differential equation (2) or its upstream collocation discretization.

4 Linear Forcing

Given the comparisons discussed in the preceding paragraph, we would like to find the optimal locations at which to evaluate the forcing function $S(x)$. (This function must be evaluated at $2m$ distinct locations in $[0, 1]$; see (4) and the discussion in Section 2.) In this context, we seek to minimize the difference between the collocation solution q_j (see (8)) and the corresponding exact solution $u(x_j)$ (see (20)). We will evaluate $S(x)$ at the points

$$\left(\frac{2p+1}{2} + \psi \pm \frac{1}{\sqrt{12}}\right) h, \tag{25}$$

$p = 0, 1, \ldots, m-1$. If $\psi = 0$, then it is clear that we are evaluating $S(x)$ at the Gauss points. Also, because the support of the basis functions f_j and g_j (see (5) and (6)) is the interval $[-\frac{1}{2}, \frac{1}{2}]$, we must restrict ψ to lie in the interval $\left[\frac{1}{\sqrt{12}} - \frac{1}{2}, \frac{1}{2} - \frac{1}{\sqrt{12}}\right]$. It is seen that $\psi \neq 0$ has the effect of moving the points at which to evaluate $S(x)$ away from the Gauss points while keeping a constant distance $\frac{h}{\sqrt{3}}$ between the two evaluation points in each subinterval. If $\psi < 0$, then we have upstreaming while $\psi > 0$ corresponds to downstreaming.

Finding the optimal locations at which to evaluate a general forcing function $S(x)$ is a formidable task. In this section, we will examine the case where $S(x)$ is a linear function, i.e., of the form

$$S(x) = \alpha_0 + \alpha_1 x, \tag{26}$$

where α_0 and α_1 are arbitrary constants.

In [3], we chose the upstream parameter ζ as a function of the Péclet number β so as to minimize (over $j = 0, 1, \ldots, m$) the maximum value of

$$\left| \frac{e^{\beta j} - 1}{e^{\beta m} - 1} - \frac{\lambda^j - 1}{\lambda^m - 1} \right|.$$

(The algorithm for doing so appears in Table 1 and is depicted in Figure 1 in Section 2 above.) In the discussion that follows, we will use this optimal value of ζ; thus the difference between $\frac{e^{\beta j}-1}{e^{\beta m}-1}$ and $\frac{\lambda^j-1}{\lambda^m-1}$ will be assumed to be already minimized.

Given $S(x)$ defined in (26), the exact solution of (2), with Dirichlet boundary conditions, evaluated at x_j, is seen to be

$$u(x_j) = \gamma_0 \alpha_0 + \gamma_1 \alpha_1 + \gamma_2, \tag{27}$$

where

$$\gamma_0 = \frac{1}{v}\left(\frac{j}{m} - \frac{e^{\beta j} - 1}{e^{\beta m} - 1}\right), \tag{28}$$

$$\gamma_1 = \frac{1}{2\beta m v}\left[(\beta j + 2)\frac{j}{m} - (\beta m + 2)\frac{e^{\beta j} - 1}{e^{\beta m} - 1}\right], \tag{29}$$

and

$$\gamma_2 = u(0) + [u(1) - u(0)]\,\frac{e^{\beta j} - 1}{e^{\beta m} - 1.} \tag{30}$$

The upstream collocation solution of the same problem is

$$q_j = \delta_0 \alpha_0 + \delta_1 \alpha_1 + \delta_2, \tag{31}$$

where

$$\delta_0 = \frac{1}{v}\left(\frac{j}{m} - \frac{\lambda^j - 1}{\lambda^m - 1}\right), \tag{32}$$

$$\delta_1 = \frac{1}{2\beta m v}\left\{[\beta j + 2(\beta\psi + \beta\zeta + 1)]\,\frac{j}{m} - [\beta m + 2(\beta\psi + \beta\zeta + 1)]\,\frac{\lambda^j - 1}{\lambda^m - 1}\right\}, \tag{33}$$

and

$$\delta_2 = u(0) + [u(1) - u(0)]\,\frac{\lambda^j - 1}{\lambda^m - 1.} \tag{34}$$

Formulas (32), (33), and (34) are obtained by using (26), evaluated at (25), in the summation in (8). The resulting summations are then simplified using the well known formula for the sum of a finite geometric series

$$\sum_{\ell=0}^{n-1} r^\ell = \frac{1 - r^n}{1 - r}$$

and the less known formula [8]

$$\sum_{\ell=1}^{n-1} \ell r^\ell = \frac{r}{(1-r)^2}\left[1 - nr^{n-1} + (n-1)r^n\right].$$

It is clear that the comparison of the exact and collocation solutions, i.e., of (27) and (31), is performed by comparing (28) to (32), (29) to (33), and (30) to (34). Note that (28) and (32) are exactly the same, except for the term $\frac{e^{\beta j}-1}{e^{\beta m}-1}$ in the former and $\frac{\lambda^j-1}{\lambda^m-1}$ in the latter. However, as discussed above, the difference between these two terms is minimized by choosing the upstream parameter ζ appropriately, and thus no further attempt is made at optimizing using (28) and (32). For exactly the same reason, we also ignore the difference between (30) to (34). Also, since the parameter of concern, namely ψ, is absent from (28), (30), (32) and (34), the goal of choosing ψ optimally is not advanced by looking at these terms.

The only place ψ appears is in (33). We compare this to (29), ignore the difference between $\frac{e^{\beta j}-1}{e^{\beta m}-1}$ and $\frac{\lambda^j-1}{\lambda^m-1}$, and thus conclude that we require

$$\psi = -\zeta. \tag{35}$$

In other words, the optimal locations at which to evaluate the forcing function (26) are precisely the points defined by optimally choosing the upstream parameter ζ. That is, to achieve optimality, we evaluate $S(x)$ and $\frac{d\widehat{u}}{dx}$ in (4) at the same points.

To conclude this section, we compare $\frac{du}{dx}(x_j)$ and its upstream collocation approximation r_j for the case of linear forcing (26). We have

$$\frac{du}{dx}(x_j) = \phi_0\alpha_0 + \phi_1\alpha_1 + \phi_2,$$

where

$$\phi_0 = \frac{1}{v}\left(1 - \frac{\beta m e^{\beta j}}{e^{\beta m} - 1}\right), \tag{36}$$

$$\phi_1 = \frac{1}{v}\left(\frac{1+\beta j}{\beta m} - \frac{2+\beta m}{2}\cdot\frac{e^{\beta j}}{e^{\beta m}-1}\right), \tag{37}$$

and

$$\phi_2 = [u(1)-u(0)]\frac{\beta m e^{\beta j}}{e^{\beta m}-1.} \tag{38}$$

Also,

$$r_j = \chi_0\alpha_0 + \chi_1\alpha_1 + \chi_2,$$

where

$$\chi_0 = \frac{1}{v}\left(1 - \frac{\beta m\lambda^j}{\lambda^m - 1}\cdot\frac{2(1+\beta\zeta)}{2+4\beta\zeta+\beta^2\zeta^2}\right), \tag{39}$$

$$\begin{aligned}\chi_1 = \quad & \frac{\beta^3\zeta^2(\psi+\zeta)+\beta^2\zeta(4\psi+5\zeta)+\beta(2\psi+6\zeta)+2+\beta j(2+4\beta\zeta+\beta^2\zeta^2)}{v\beta m(2+4\beta\zeta+\beta^2\zeta^2)} \\ & -\lambda^j\frac{2\beta^2\zeta(\psi+\zeta)+2+\beta m+\beta^2 m\zeta+\beta(2\psi+4\zeta)}{v(\lambda^m-1)(2+4\beta\zeta+\beta^2\zeta^2),}\end{aligned} \tag{40}$$

and

$$\chi_2 = [u(1)-u(0)]\frac{\beta m\lambda^j}{\lambda^m-1}\cdot\frac{2(1+\beta\zeta)}{2+4\beta\zeta+\beta^2\zeta^2.} \tag{41}$$

Note that if $\zeta = 0$, then we have the now familiar correspondence between ϕ_0 and χ_0 (see (36) and (39)) and between ϕ_2 and χ_2 (see (38) and (41)). Furthermore, if we choose ψ optimally via (35), then (40) reduces to

$$\chi_1 = \frac{1}{v}\left[\frac{1+\beta j}{\beta m} - \frac{2+\beta m}{2+4\beta\zeta+\beta^2\zeta^2}\cdot\frac{\lambda^j}{\lambda^m-1}\cdot(1+\beta\zeta)\right],$$

which, if $\zeta = 0$, possesses the familiar correspondence with ϕ_1 (see (37).

5 Numerical Experiments

In this section, we give results of numerical experiments that illustrate the theory presented above. We will solve (2) where $S(x)$ is the linear function $S(x) = 3 - 2x$ (Problem A) and $S(x) = -10+200x$ (Problem B). The boundary conditions for both problems are $u(0) = 1$ and $u(1) = 0$. The number of subintervals is fixed at $m = 10$, as is the convection coefficient $v = 10$. The diffusion coefficient D assumes values $D = 1.0, 0.25, 0.1, 0.05, 0.02$, thus producing Péclet numbers $\beta = 1, 4, 10, 20, 50$, respectively. For a given value of β, we use the corresponding optimal value of ζ as determined from Table 1. The parameter ψ, which controls where we evaluate the forcing function $S(x)$, takes on the values

$$\psi = 0.20, 0.18, 0.16, \ldots, -0.20, -\zeta. \tag{42}$$

Recall that this last value of ψ is considered to be optimal. The results are illustrated in Figures 2 through 6 for Problem A and in Figures 7 through 11 for Problem B.

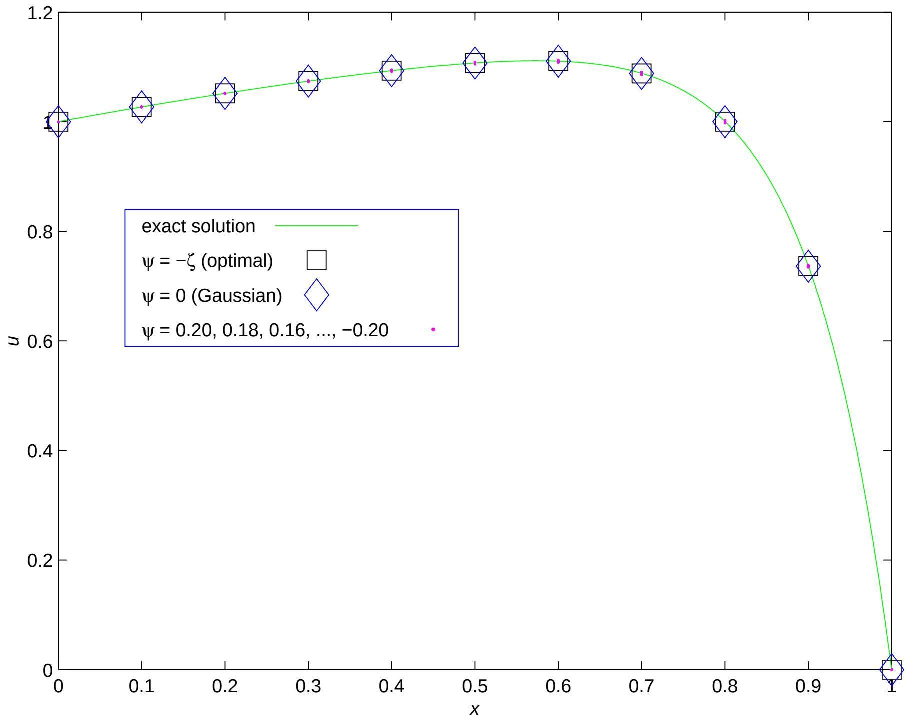

Figure 2: Comparison of exact and various collocation solutions: $\beta = 1$, Problem A

The symbols used in each of these figures are the same. Let us initially take Figure 9 as an example. The horizontal axis represents the spatial variable x while the vertical axis represents u, the solution of the boundary value problem (BVP). The exact solution of the BVP is given by the continuous curve. The optimal (i.e., $\psi = -\zeta$) solution is depicted by squares. The solution obtained using $\psi = 0$ (i.e., evaluating $S(x)$ at the Gauss points) is depicted by non-square rhombi. Furthermore, we depict using dots the solution obtained using the values of ψ in (42). Because we use dots to indicate each of these values of ψ, it is difficult to discern how the collocation solution changes as we vary ψ. We thus compile our data in Appendix A in Tables 2 through 11, which correspond to Figures 2 through 11, respectively.

Let us note some features of Figure 9. First, the optimal collocation solution is visually indistinguishable from the exact solution. Next, there are significant differences between the exact solution and the Gaussian collocation solution, particularly for larger values of x. Finally, as seen by the vertical distribution of dots, we note that varying ψ can have a significant effect on the solution obtained, again particularly at the larger values of x.

Let us now compare Figures 2 through 6, each of which represents solving Problem A using a different Péclet number. As the Péclet number increases, the front that exists in the solution near $x = 1$ becomes more pronounced. However, regard-

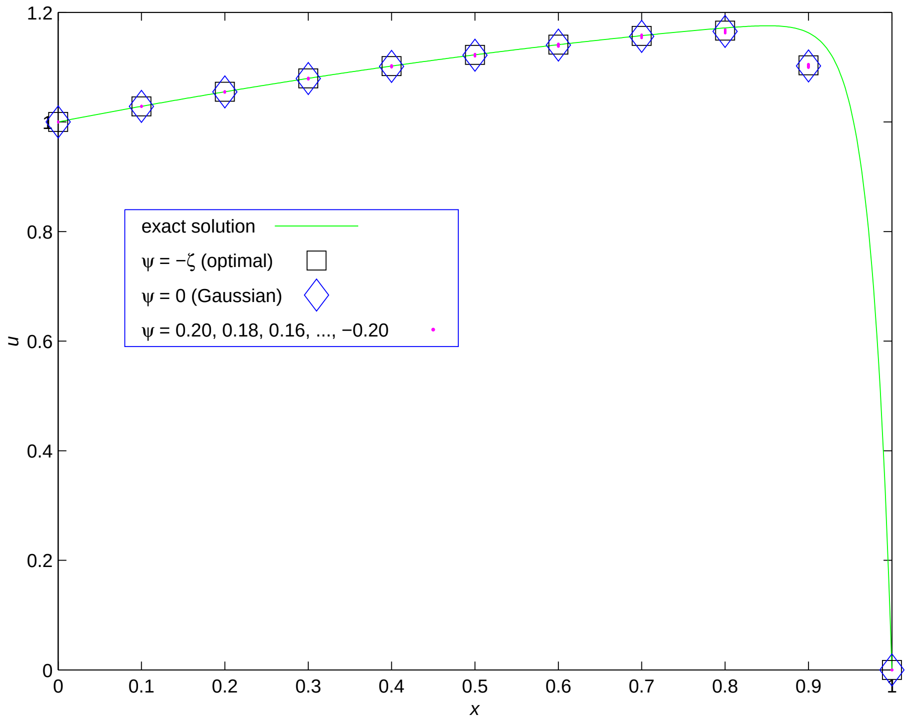

Figure 3: Comparison of exact and various collocation solutions: $\beta = 4$, Problem A

less of the steepness of this front, varying ψ has almost no effect on the solution, as seen from the very tight vertical distribution of dots. We see for $\beta = 4$ in Figure 3 that no collocation solution accurately captures the behavior of the exact solution at $x = 0.9$. This is not due to the particular forcing function chosen, as similar behavior is observed in Figure 8 and was observed in [3] for the case of $S(x) = 0$.

Now we focus on Problem B by comparing Figures 7 through 11. Here we are using the same Péclet numbers used to solve Problem A. The behavior here is quite different in some respects when compared to Problem A. First, we note the wide vertical distribution of dots in these figures, showing that varying ψ can significantly affect the solution obtained. This is no surprise as the absolute value of α_1 is much larger in Problem B than it is in Problem A. More significant is the fact that for larger Péclet numbers β, there is visually no difference between the exact solution and the optimal collocation solution (this was true for Problem A as well). This can be explained as follows.

When β is large, so is λ; so the fractions $\frac{e^{\beta j}-1}{e^{\beta m}-1}$ and $\frac{\lambda^j-1}{\lambda^m-1}$ (which appear in (28), (29), (30), (32), (33), and (34)) are both approximately zero for $j = 1, 2, 3, \ldots, m-1$. If these fractions are set equal to zero and we take (35) in (33), then there is no difference between (28) and (32), between (29) and (33), and between (30) and (34). Thus for large β, we have that (27) (the exact solution) and (31) (the numerical solution) are approximately equal.

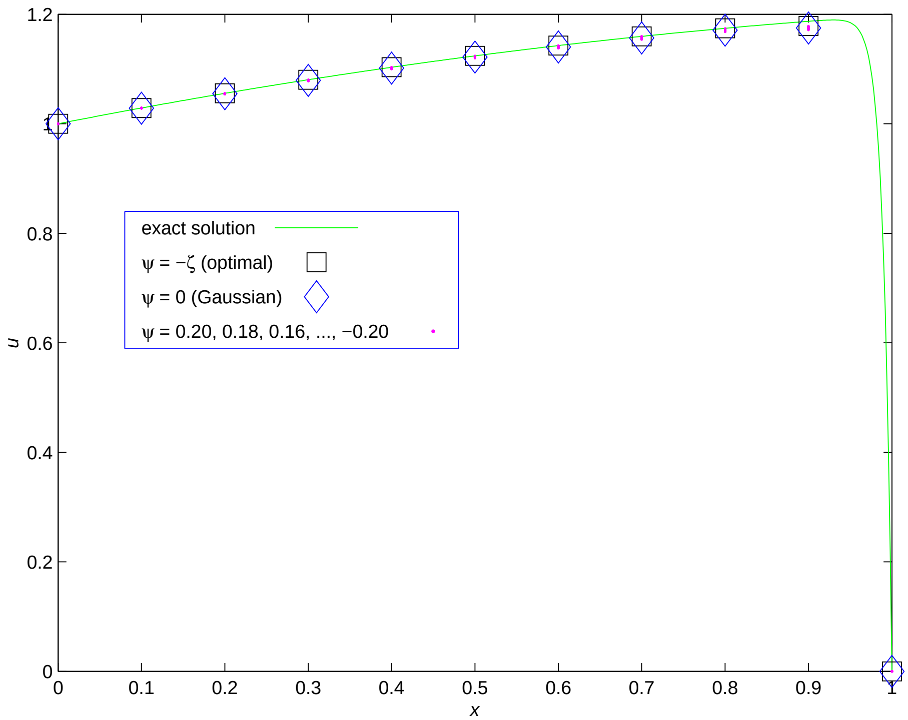

Figure 4: Comparison of exact and various collocation solutions: $\beta = 10$, Problem A

For a slightly different reason, for small Péclet numbers, (27) and (31) are also approximately equal. As seen in [2], $\lambda \approx e^{\beta}$ for $0 < \beta \leq 1.5$ (as mentioned above in Section 3, the McLaurin series expansions of λ (when $\zeta = 0$) and e^{β} agree for the first five terms; again see [2]). Taking $\lambda = e^{\beta}$ and $\zeta = 0$ (which is optimal for small β), we have that (28) and (32) are equal, as are (29) and (33), and (30) and (34).

6 Summary and Conclusions

In this paper, we give analytic formulas for the solution of the Hermite collocation discretization of the one-dimensional constant-coefficient nonhomogeneous convection-diffusion equation, defined on a uniform mesh with Dirichlet boundary conditions. Upstream weighting is employed in the evaluation of the derivative of the convective term.

There are two free parameters: the upstreaming parameter ζ and the parameter ψ that governs where to evaluate the forcing function. The former is chosen optimally as described in [3]. The latter, for the case where the forcing function is linear, has optimal value $\psi = -\zeta$; that is, we evaluate both the convective term and the forcing function at the optimal locations given in [3]. Numerical experiments show excellent agreement between the exact solution of boundary value problems and their optimal collocation approximations, particularly for small (≤ 1.5) and

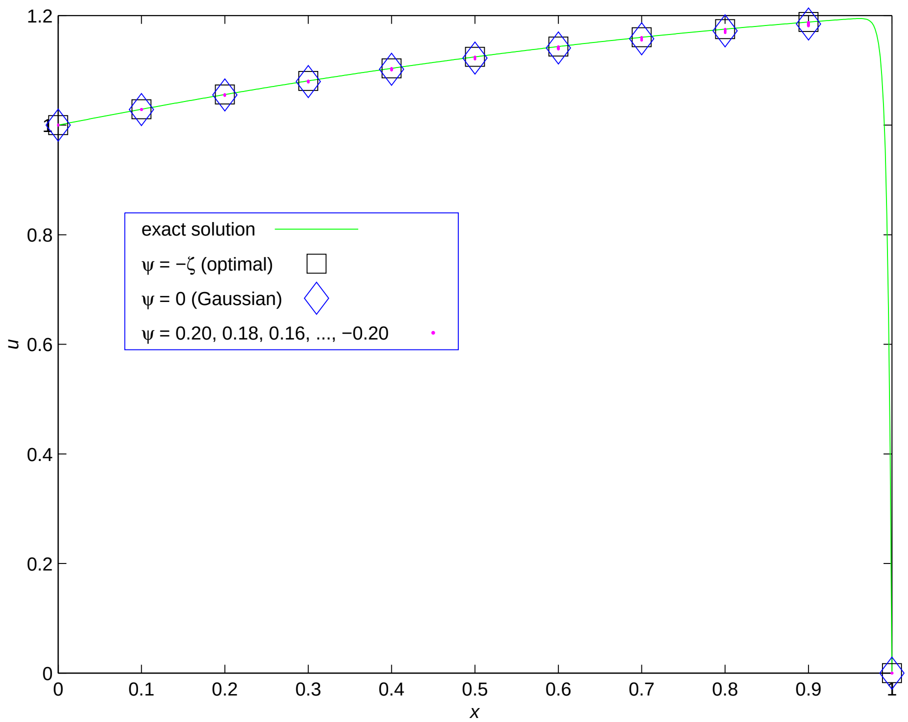

Figure 5: Comparison of exact and various collocation solutions: $\beta = 20$, Problem A

large (≥ 10) Péclet numbers.

A Data from Numerical Experiments.

In this appendix we display the data used to produce Figures 2 through 11. These data appear in Tables 2 through 11, respectively.

In each of these tables, the first row gives the values of x at which the exact and numerical solutions are computed. The second row gives the exact solution at these values of x. The remainder of the rows give the collocation solutions corresponding to the value of ψ that appears in the first column, with the third row providing the optimal $\psi = -\zeta$ collocation solution.

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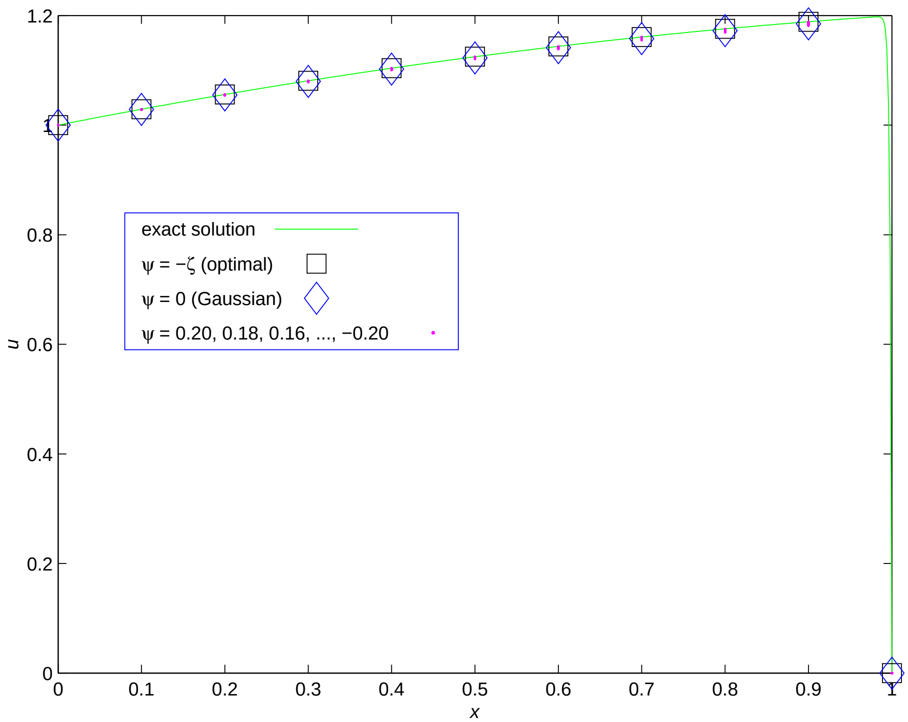

Figure 6: Comparison of exact and various collocation solutions: $\beta = 50$, Problem A

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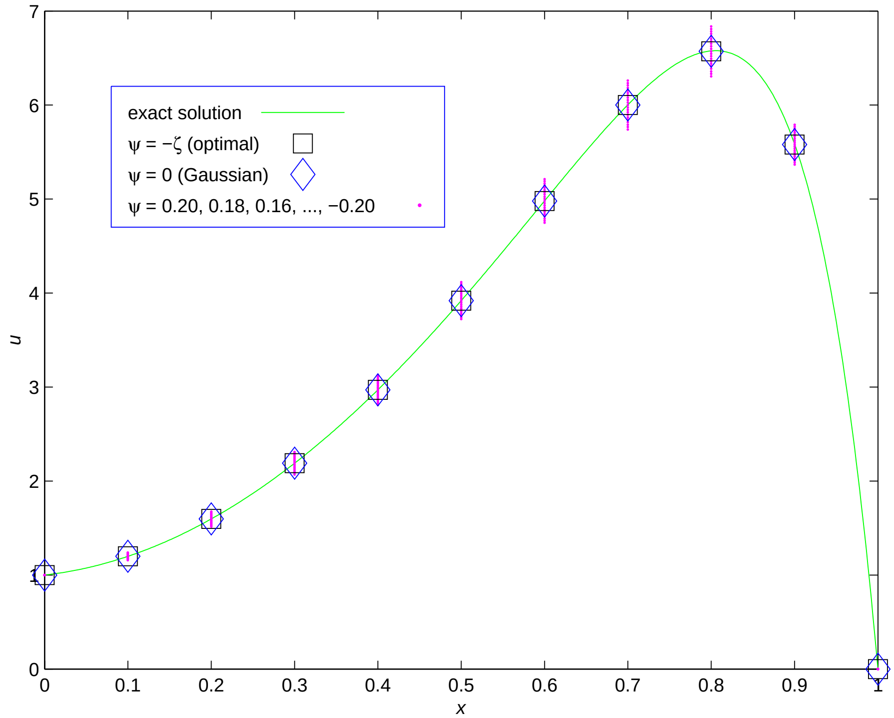

Figure 7: Comparison of exact and various collocation solutions: $\beta = 1$, Problem B

Table 2: Data for $\beta = 1$, Problem A (see Figure 2).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.027	1.052	1.074	1.093	1.107	1.110	1.088	1.000	0.737	0.000
0.00	1.000	1.027	1.052	1.074	1.093	1.107	1.110	1.088	1.000	0.736	0.000
-0.20	1.000	1.027	1.052	1.075	1.095	1.109	1.113	1.091	1.003	0.738	0.000
-0.18	1.000	1.027	1.052	1.075	1.095	1.109	1.112	1.090	1.002	0.738	0.000
-0.16	1.000	1.027	1.052	1.075	1.094	1.109	1.112	1.090	1.002	0.738	0.000
-0.14	1.000	1.027	1.052	1.075	1.094	1.108	1.112	1.090	1.002	0.738	0.000
-0.12	1.000	1.027	1.052	1.075	1.094	1.108	1.112	1.090	1.001	0.738	0.000
-0.10	1.000	1.027	1.052	1.075	1.094	1.108	1.111	1.089	1.001	0.737	0.000
-0.08	1.000	1.027	1.052	1.074	1.094	1.108	1.111	1.089	1.001	0.737	0.000
-0.06	1.000	1.027	1.052	1.074	1.094	1.108	1.111	1.089	1.001	0.737	0.000
-0.04	1.000	1.027	1.052	1.074	1.093	1.107	1.111	1.089	1.000	0.737	0.000
-0.02	1.000	1.027	1.052	1.074	1.093	1.107	1.111	1.088	1.000	0.737	0.000
0.00	1.000	1.027	1.052	1.074	1.093	1.107	1.110	1.088	1.000	0.736	0.000
0.02	1.000	1.027	1.052	1.074	1.093	1.107	1.110	1.088	1.000	0.736	0.000
0.04	1.000	1.027	1.051	1.074	1.093	1.107	1.110	1.088	0.999	0.736	0.000
0.06	1.000	1.027	1.051	1.074	1.093	1.106	1.110	1.087	0.999	0.736	0.000
0.08	1.000	1.027	1.051	1.073	1.092	1.106	1.109	1.087	0.999	0.735	0.000
0.10	1.000	1.027	1.051	1.073	1.092	1.106	1.109	1.087	0.999	0.735	0.000
0.12	1.000	1.027	1.051	1.073	1.092	1.106	1.109	1.086	0.998	0.735	0.000
0.14	1.000	1.027	1.051	1.073	1.092	1.106	1.109	1.086	0.998	0.735	0.000
0.16	1.000	1.027	1.051	1.073	1.092	1.105	1.108	1.086	0.998	0.735	0.000
0.18	1.000	1.027	1.051	1.073	1.092	1.105	1.108	1.086	0.997	0.734	0.000
0.20	1.000	1.027	1.051	1.073	1.092	1.105	1.108	1.085	0.997	0.734	0.000

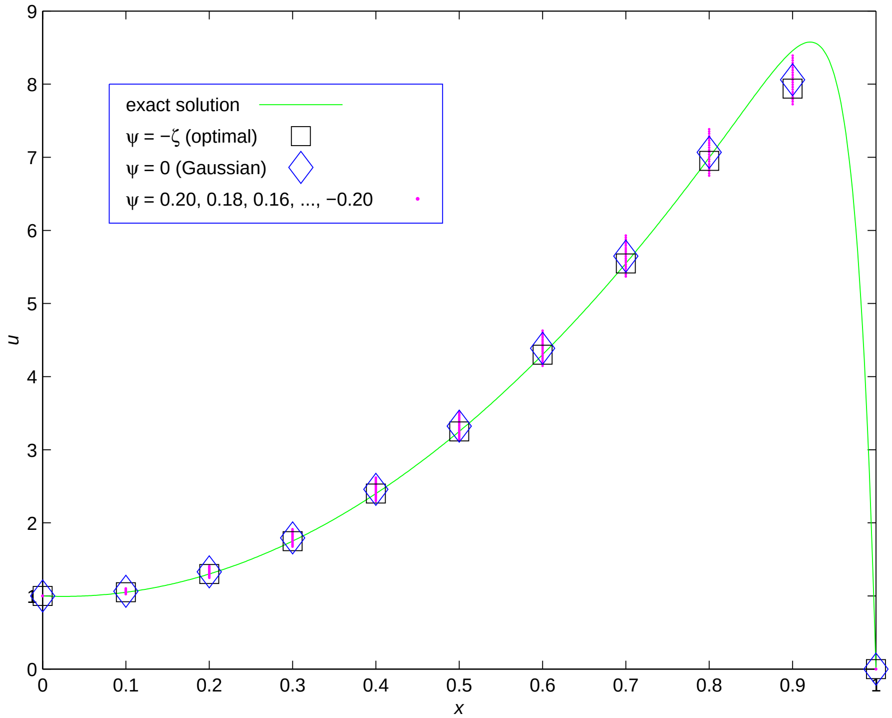

Figure 8: Comparison of exact and various collocation solutions: $\beta = 4$, Problem B

Table 3: Data for $\beta = 4$, Problem A (see Figure 3).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.028	1.055	1.079	1.102	1.123	1.141	1.157	1.172	1.163	0.000
-0.07	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.167	1.103	0.000
-0.20	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.159	1.169	1.106	0.000
-0.18	1.000	1.029	1.055	1.080	1.103	1.124	1.142	1.159	1.168	1.105	0.000
-0.16	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.158	1.168	1.105	0.000
-0.14	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.158	1.168	1.105	0.000
-0.12	1.000	1.029	1.055	1.080	1.102	1.123	1.142	1.158	1.167	1.104	0.000
-0.10	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.158	1.167	1.104	0.000
-0.08	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.157	1.167	1.104	0.000
-0.06	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.166	1.103	0.000
-0.04	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.166	1.103	0.000
-0.02	1.000	1.028	1.055	1.079	1.102	1.122	1.140	1.156	1.166	1.103	0.000
0.00	1.000	1.028	1.055	1.079	1.101	1.122	1.140	1.156	1.165	1.102	0.000
0.02	1.000	1.028	1.055	1.079	1.101	1.122	1.140	1.156	1.165	1.102	0.000
0.04	1.000	1.028	1.055	1.079	1.101	1.121	1.140	1.156	1.165	1.102	0.000
0.06	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.155	1.164	1.101	0.000
0.08	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.155	1.164	1.101	0.000
0.10	1.000	1.028	1.054	1.078	1.101	1.121	1.139	1.155	1.164	1.101	0.000
0.12	1.000	1.028	1.054	1.078	1.100	1.121	1.139	1.154	1.163	1.100	0.000
0.14	1.000	1.028	1.054	1.078	1.100	1.120	1.138	1.154	1.163	1.100	0.000
0.16	1.000	1.028	1.054	1.078	1.100	1.120	1.138	1.154	1.163	1.100	0.000
0.18	1.000	1.028	1.054	1.078	1.100	1.120	1.138	1.154	1.162	1.099	0.000
0.20	1.000	1.028	1.054	1.078	1.100	1.120	1.138	1.153	1.162	1.099	0.000

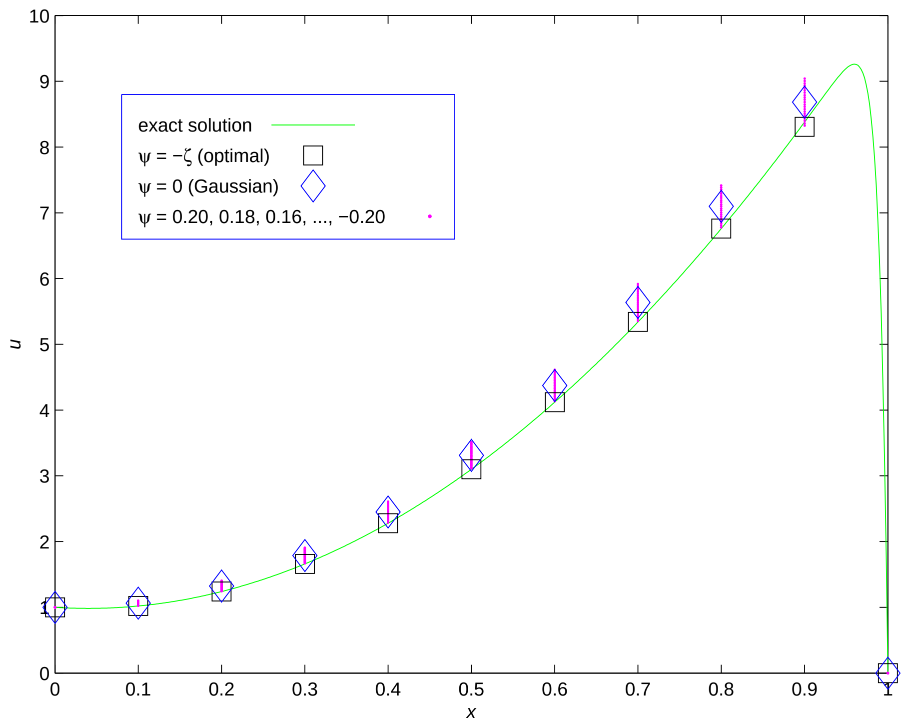

Figure 9: Comparison of exact and various collocation solutions: $\beta = 10$, Problem B

Table 4: Data for $\beta = 10$, Problem A (see Figure 4).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.160	1.174	1.187	0.000
-0.21	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.160	1.174	1.179	0.000
-0.20	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.159	1.174	1.178	0.000
-0.18	1.000	1.029	1.055	1.080	1.103	1.124	1.142	1.159	1.174	1.178	0.000
-0.16	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.159	1.174	1.178	0.000
-0.14	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.159	1.173	1.177	0.000
-0.12	1.000	1.029	1.055	1.080	1.102	1.123	1.142	1.158	1.173	1.177	0.000
-0.10	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.158	1.173	1.177	0.000
-0.08	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.158	1.172	1.176	0.000
-0.06	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.172	1.176	0.000
-0.04	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.172	1.175	0.000
-0.02	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.171	1.175	0.000
0.00	1.000	1.028	1.055	1.079	1.102	1.122	1.140	1.157	1.171	1.175	0.000
0.02	1.000	1.028	1.055	1.079	1.101	1.122	1.140	1.156	1.171	1.174	0.000
0.04	1.000	1.028	1.055	1.079	1.101	1.121	1.140	1.156	1.170	1.174	0.000
0.06	1.000	1.028	1.055	1.079	1.101	1.121	1.140	1.156	1.170	1.174	0.000
0.08	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.156	1.170	1.173	0.000
0.10	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.155	1.169	1.173	0.000
0.12	1.000	1.028	1.054	1.078	1.101	1.121	1.139	1.155	1.169	1.173	0.000
0.14	1.000	1.028	1.054	1.078	1.100	1.120	1.139	1.155	1.169	1.172	0.000
0.16	1.000	1.028	1.054	1.078	1.100	1.120	1.138	1.154	1.168	1.172	0.000
0.18	1.000	1.028	1.054	1.078	1.100	1.120	1.138	1.154	1.168	1.172	0.000
0.20	1.000	1.028	1.054	1.078	1.100	1.120	1.138	1.154	1.168	1.171	0.000

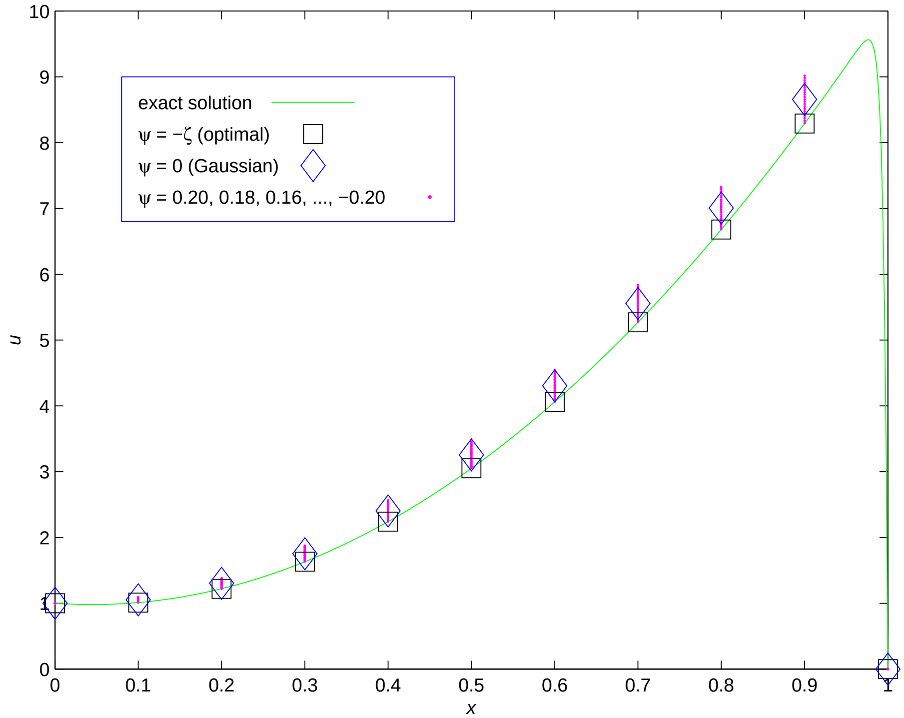

Figure 10: Comparison of exact and various collocation solutions: $\beta = 20$, Problem B

Table 5: Data for $\beta = 20$, Problem A (see Figure 5).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.029	1.056	1.081	1.104	1.125	1.143	1.160	1.175	1.188	0.000
-0.20	1.000	1.029	1.056	1.081	1.104	1.125	1.143	1.160	1.175	1.188	0.000
-0.20	1.000	1.029	1.056	1.081	1.104	1.124	1.143	1.160	1.175	1.188	0.000
-0.18	1.000	1.029	1.056	1.081	1.103	1.124	1.143	1.160	1.175	1.188	0.000
-0.16	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.160	1.174	1.187	0.000
-0.14	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.159	1.174	1.187	0.000
-0.12	1.000	1.029	1.055	1.080	1.103	1.124	1.142	1.159	1.174	1.187	0.000
-0.10	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.159	1.174	1.186	0.000
-0.08	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.159	1.173	1.186	0.000
-0.06	1.000	1.029	1.055	1.080	1.102	1.123	1.142	1.158	1.173	1.186	0.000
-0.04	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.158	1.173	1.185	0.000
-0.02	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.158	1.172	1.185	0.000
0.00	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.172	1.184	0.000
0.02	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.172	1.184	0.000
0.04	1.000	1.028	1.055	1.079	1.102	1.122	1.140	1.157	1.171	1.184	0.000
0.06	1.000	1.028	1.055	1.079	1.101	1.122	1.140	1.157	1.171	1.183	0.000
0.08	1.000	1.028	1.055	1.079	1.101	1.122	1.140	1.156	1.171	1.183	0.000
0.10	1.000	1.028	1.055	1.079	1.101	1.121	1.140	1.156	1.170	1.183	0.000
0.12	1.000	1.028	1.055	1.079	1.101	1.121	1.140	1.156	1.170	1.182	0.000
0.14	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.155	1.170	1.182	0.000
0.16	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.155	1.169	1.182	0.000
0.18	1.000	1.028	1.054	1.078	1.101	1.121	1.139	1.155	1.169	1.181	0.000
0.20	1.000	1.028	1.054	1.078	1.100	1.120	1.139	1.155	1.169	1.181	0.000

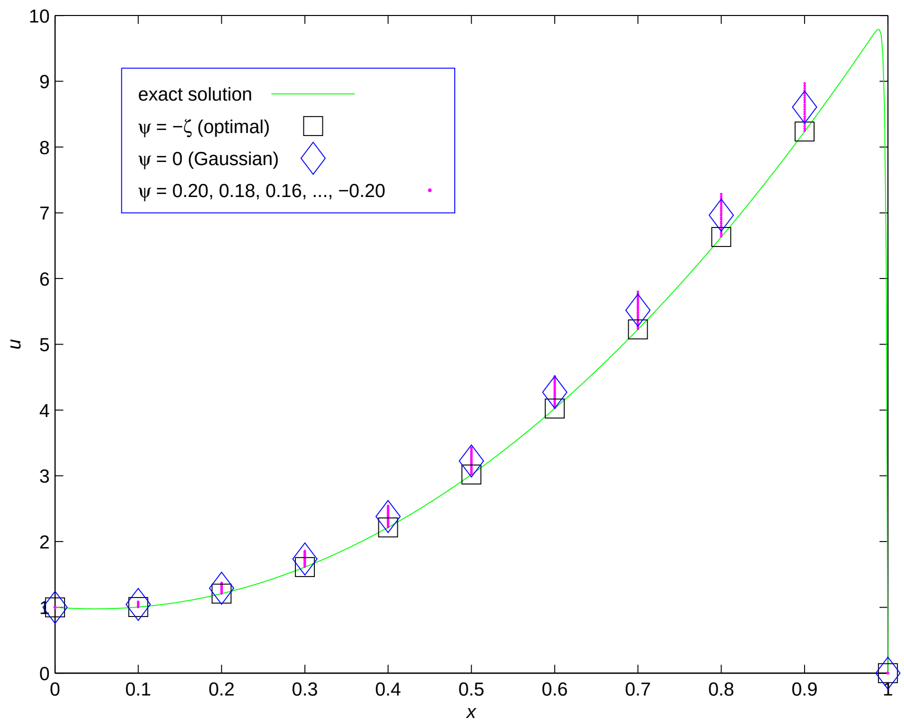

Figure 11: Comparison of exact and various collocation solutions: $\beta = 50$, Problem B

Table 6: Data for $\beta = 50$, Problem A (see Figure 6).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.029	1.056	1.081	1.104	1.125	1.144	1.161	1.176	1.189	0.000
-0.21	1.000	1.029	1.056	1.081	1.104	1.125	1.144	1.161	1.176	1.189	0.000
-0.20	1.000	1.029	1.056	1.081	1.104	1.125	1.144	1.161	1.176	1.189	0.000
-0.18	1.000	1.029	1.056	1.081	1.104	1.125	1.143	1.160	1.175	1.188	0.000
-0.16	1.000	1.029	1.056	1.081	1.103	1.124	1.143	1.160	1.175	1.188	0.000
-0.14	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.160	1.175	1.187	0.000
-0.12	1.000	1.029	1.056	1.080	1.103	1.124	1.143	1.160	1.174	1.187	0.000
-0.10	1.000	1.029	1.055	1.080	1.103	1.124	1.142	1.159	1.174	1.187	0.000
-0.08	1.000	1.029	1.055	1.080	1.103	1.124	1.142	1.159	1.174	1.186	0.000
-0.06	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.159	1.173	1.186	0.000
-0.04	1.000	1.029	1.055	1.080	1.103	1.123	1.142	1.158	1.173	1.186	0.000
-0.02	1.000	1.029	1.055	1.080	1.102	1.123	1.142	1.158	1.173	1.185	0.000
0.00	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.158	1.172	1.185	0.000
0.02	1.000	1.029	1.055	1.080	1.102	1.123	1.141	1.158	1.172	1.185	0.000
0.04	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.172	1.184	0.000
0.06	1.000	1.028	1.055	1.079	1.102	1.122	1.141	1.157	1.171	1.184	0.000
0.08	1.000	1.028	1.055	1.079	1.102	1.122	1.140	1.157	1.171	1.183	0.000
0.10	1.000	1.028	1.055	1.079	1.101	1.122	1.140	1.156	1.171	1.183	0.000
0.12	1.000	1.028	1.055	1.079	1.101	1.122	1.140	1.156	1.170	1.183	0.000
0.14	1.000	1.028	1.055	1.079	1.101	1.121	1.140	1.156	1.170	1.182	0.000
0.16	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.156	1.170	1.182	0.000
0.18	1.000	1.028	1.054	1.079	1.101	1.121	1.139	1.155	1.169	1.182	0.000
0.20	1.000	1.028	1.054	1.078	1.101	1.121	1.139	1.155	1.169	1.181	0.000

Table 7: Data for $\beta = 1$, Problem B (see Figure 7).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.199	1.597	2.190	2.971	3.920	4.981	6.003	6.576	5.586	0.000
0.00	1.000	1.199	1.596	2.189	2.971	3.919	4.979	6.000	6.572	5.579	0.000
-0.20	1.000	1.159	1.517	2.070	2.812	3.722	4.747	5.740	6.306	5.367	0.000
-0.18	1.000	1.163	1.525	2.082	2.827	3.742	4.770	5.766	6.333	5.388	0.000
-0.16	1.000	1.167	1.533	2.094	2.843	3.761	4.793	5.792	6.359	5.409	0.000
-0.14	1.000	1.171	1.541	2.106	2.859	3.781	4.817	5.818	6.386	5.430	0.000
-0.12	1.000	1.175	1.549	2.118	2.875	3.801	4.840	5.844	6.412	5.452	0.000
-0.10	1.000	1.179	1.557	2.130	2.891	3.820	4.863	5.870	6.439	5.473	0.000
-0.08	1.000	1.183	1.565	2.142	2.907	3.840	4.886	5.896	6.465	5.494	0.000
-0.06	1.000	1.187	1.573	2.154	2.923	3.860	4.910	5.922	6.492	5.516	0.000
-0.04	1.000	1.191	1.581	2.166	2.939	3.880	4.933	5.948	6.519	5.537	0.000
-0.02	1.000	1.195	1.588	2.178	2.955	3.899	4.956	5.974	6.545	5.558	0.000
0.00	1.000	1.199	1.596	2.189	2.971	3.919	4.979	6.000	6.572	5.579	0.000
0.02	1.000	1.203	1.604	2.201	2.986	3.939	5.003	6.026	6.598	5.601	0.000
0.04	1.000	1.207	1.612	2.213	3.002	3.959	5.026	6.052	6.625	5.622	0.000
0.06	1.000	1.211	1.620	2.225	3.018	3.978	5.049	6.078	6.651	5.643	0.000
0.08	1.000	1.215	1.628	2.237	3.034	3.998	5.073	6.104	6.678	5.664	0.000
0.10	1.000	1.219	1.636	2.249	3.050	4.018	5.096	6.130	6.705	5.686	0.000
0.12	1.000	1.223	1.644	2.261	3.066	4.037	5.119	6.156	6.731	5.707	0.000
0.14	1.000	1.227	1.652	2.273	3.082	4.057	5.142	6.182	6.758	5.728	0.000
0.16	1.000	1.231	1.660	2.285	3.098	4.077	5.166	6.208	6.784	5.749	0.000
0.18	1.000	1.235	1.668	2.297	3.114	4.097	5.189	6.234	6.811	5.771	0.000
0.20	1.000	1.239	1.676	2.309	3.130	4.116	5.212	6.260	6.837	5.792	0.000

Table 8: Data for $\beta = 4$, Problem B (see Figure 8).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.050	1.300	1.750	2.400	3.250	4.300	5.550	6.996	8.458	0.000
-0.07	1.000	1.050	1.300	1.750	2.400	3.250	4.300	5.547	6.952	7.938	0.000
-0.20	1.000	1.025	1.249	1.674	2.298	3.123	4.147	5.369	6.749	7.726	0.000
-0.18	1.000	1.029	1.257	1.686	2.314	3.143	4.171	5.397	6.781	7.759	0.000
-0.16	1.000	1.033	1.265	1.698	2.330	3.163	4.195	5.425	6.813	7.793	0.000
-0.14	1.000	1.037	1.273	1.710	2.346	3.183	4.219	5.453	6.845	7.826	0.000
-0.12	1.000	1.041	1.281	1.722	2.362	3.203	4.243	5.481	6.877	7.859	0.000
-0.10	1.000	1.045	1.289	1.734	2.378	3.223	4.267	5.509	6.908	7.893	0.000
-0.08	1.000	1.049	1.297	1.746	2.394	3.243	4.291	5.537	6.940	7.926	0.000
-0.06	1.000	1.053	1.305	1.758	2.410	3.263	4.315	5.565	6.972	7.959	0.000
-0.04	1.000	1.057	1.313	1.770	2.426	3.283	4.339	5.593	7.004	7.992	0.000
-0.02	1.000	1.061	1.321	1.782	2.442	3.303	4.363	5.621	7.036	8.026	0.000
0.00	1.000	1.065	1.329	1.794	2.458	3.323	4.387	5.649	7.067	8.059	0.000
0.02	1.000	1.069	1.337	1.806	2.474	3.343	4.411	5.677	7.099	8.092	0.000
0.04	1.000	1.073	1.345	1.818	2.490	3.363	4.435	5.705	7.131	8.126	0.000
0.06	1.000	1.077	1.353	1.830	2.506	3.383	4.459	5.732	7.163	8.159	0.000
0.08	1.000	1.081	1.361	1.842	2.522	3.403	4.483	5.760	7.195	8.192	0.000
0.10	1.000	1.085	1.369	1.854	2.538	3.423	4.507	5.788	7.227	8.225	0.000
0.12	1.000	1.089	1.377	1.866	2.554	3.443	4.531	5.816	7.258	8.259	0.000
0.14	1.000	1.093	1.385	1.878	2.570	3.463	4.555	5.844	7.290	8.292	0.000
0.16	1.000	1.097	1.393	1.890	2.586	3.483	4.579	5.872	7.322	8.325	0.000
0.18	1.000	1.101	1.401	1.902	2.602	3.503	4.603	5.900	7.354	8.359	0.000
0.20	1.000	1.105	1.409	1.914	2.618	3.523	4.627	5.928	7.386	8.392	0.000

Table 9: Data for $\beta = 10$, Problem B (see Figure 9).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.020	1.240	1.660	2.280	3.100	4.120	5.340	6.760	8.380	0.000
-0.21	1.000	1.020	1.240	1.660	2.280	3.100	4.120	5.340	6.759	8.306	0.000
-0.20	1.000	1.022	1.245	1.667	2.289	3.111	4.134	5.356	6.778	8.327	0.000
-0.18	1.000	1.026	1.253	1.679	2.305	3.131	4.158	5.384	6.810	8.362	0.000
-0.16	1.000	1.030	1.261	1.691	2.321	3.151	4.182	5.412	6.842	8.398	0.000
-0.14	1.000	1.034	1.269	1.703	2.337	3.171	4.206	5.440	6.874	8.434	0.000
-0.12	1.000	1.038	1.277	1.715	2.353	3.191	4.230	5.468	6.906	8.469	0.000
-0.10	1.000	1.042	1.285	1.727	2.369	3.211	4.254	5.496	6.938	8.505	0.000
-0.08	1.000	1.046	1.293	1.739	2.385	3.231	4.278	5.524	6.970	8.541	0.000
-0.06	1.000	1.050	1.301	1.751	2.401	3.251	4.302	5.552	7.002	8.577	0.000
-0.04	1.000	1.054	1.309	1.763	2.417	3.271	4.326	5.580	7.034	8.612	0.000
-0.02	1.000	1.058	1.317	1.775	2.433	3.291	4.350	5.608	7.066	8.648	0.000
0.00	1.000	1.062	1.325	1.787	2.449	3.311	4.374	5.636	7.098	8.684	0.000
0.02	1.000	1.066	1.333	1.799	2.465	3.331	4.398	5.664	7.130	8.719	0.000
0.04	1.000	1.070	1.341	1.811	2.481	3.351	4.422	5.692	7.162	8.755	0.000
0.06	1.000	1.074	1.349	1.823	2.497	3.371	4.446	5.720	7.194	8.791	0.000
0.08	1.000	1.078	1.357	1.835	2.513	3.391	4.470	5.748	7.226	8.827	0.000
0.10	1.000	1.082	1.365	1.847	2.529	3.411	4.494	5.776	7.258	8.862	0.000
0.12	1.000	1.086	1.373	1.859	2.545	3.431	4.518	5.804	7.290	8.898	0.000
0.14	1.000	1.090	1.381	1.871	2.561	3.451	4.542	5.832	7.322	8.934	0.000
0.16	1.000	1.094	1.389	1.883	2.577	3.471	4.566	5.860	7.354	8.969	0.000
0.18	1.000	1.098	1.397	1.895	2.593	3.491	4.590	5.888	7.386	9.005	0.000
0.20	1.000	1.102	1.405	1.907	2.609	3.511	4.614	5.916	7.418	9.041	0.000

Table 10: Data for $\beta = 20$, Problem B (see Figure 10).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.010	1.220	1.630	2.240	3.050	4.060	5.270	6.680	8.290	0.000
-0.20	1.000	1.010	1.220	1.630	2.240	3.050	4.060	5.270	6.680	8.290	0.000
-0.20	1.000	1.011	1.222	1.633	2.243	3.054	4.065	5.276	6.687	8.298	0.000
-0.18	1.000	1.015	1.230	1.645	2.259	3.074	4.089	5.304	6.719	8.334	0.000
-0.16	1.000	1.019	1.238	1.657	2.275	3.094	4.113	5.332	6.751	8.370	0.000
-0.14	1.000	1.023	1.246	1.669	2.291	3.114	4.137	5.360	6.783	8.406	0.000
-0.12	1.000	1.027	1.254	1.681	2.307	3.134	4.161	5.388	6.815	8.442	0.000
-0.10	1.000	1.031	1.262	1.693	2.323	3.154	4.185	5.416	6.847	8.478	0.000
-0.08	1.000	1.035	1.270	1.705	2.339	3.174	4.209	5.444	6.879	8.514	0.000
-0.06	1.000	1.039	1.278	1.717	2.355	3.194	4.233	5.472	6.911	8.550	0.000
-0.04	1.000	1.043	1.286	1.729	2.371	3.214	4.257	5.500	6.943	8.586	0.000
-0.02	1.000	1.047	1.294	1.741	2.387	3.234	4.281	5.528	6.975	8.622	0.000
0.00	1.000	1.051	1.302	1.753	2.403	3.254	4.305	5.556	7.007	8.658	0.000
0.02	1.000	1.055	1.310	1.765	2.419	3.274	4.329	5.584	7.039	8.694	0.000
0.04	1.000	1.059	1.318	1.777	2.435	3.294	4.353	5.612	7.071	8.730	0.000
0.06	1.000	1.063	1.326	1.789	2.451	3.314	4.377	5.640	7.103	8.766	0.000
0.08	1.000	1.067	1.334	1.801	2.467	3.334	4.401	5.668	7.135	8.802	0.000
0.10	1.000	1.071	1.342	1.813	2.483	3.354	4.425	5.696	7.167	8.838	0.000
0.12	1.000	1.075	1.350	1.825	2.499	3.374	4.449	5.724	7.199	8.874	0.000
0.14	1.000	1.079	1.358	1.837	2.515	3.394	4.473	5.752	7.231	8.910	0.000
0.16	1.000	1.083	1.366	1.849	2.531	3.414	4.497	5.780	7.263	8.946	0.000
0.18	1.000	1.087	1.374	1.861	2.547	3.434	4.521	5.808	7.295	8.982	0.000
0.20	1.000	1.091	1.382	1.873	2.563	3.454	4.545	5.836	7.327	9.018	0.000

Table 11: Data for $\beta = 50$, Problem B (see Figure 11).

	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
	1.000	1.004	1.208	1.612	2.216	3.020	4.024	5.228	6.632	8.236	0.000
-0.21	1.000	1.004	1.208	1.612	2.216	3.020	4.024	5.228	6.632	8.236	0.000
-0.20	1.000	1.005	1.211	1.616	2.221	3.027	4.032	5.237	6.643	8.248	0.000
-0.18	1.000	1.009	1.219	1.628	2.237	3.047	4.056	5.265	6.675	8.284	0.000
-0.16	1.000	1.013	1.227	1.640	2.253	3.067	4.080	5.293	6.707	8.320	0.000
-0.14	1.000	1.017	1.235	1.652	2.269	3.087	4.104	5.321	6.739	8.356	0.000
-0.12	1.000	1.021	1.243	1.664	2.285	3.107	4.128	5.349	6.771	8.392	0.000
-0.10	1.000	1.025	1.251	1.676	2.301	3.127	4.152	5.377	6.803	8.428	0.000
-0.08	1.000	1.029	1.259	1.688	2.317	3.147	4.176	5.405	6.835	8.464	0.000
-0.06	1.000	1.033	1.267	1.700	2.333	3.167	4.200	5.433	6.867	8.500	0.000
-0.04	1.000	1.037	1.275	1.712	2.349	3.187	4.224	5.461	6.899	8.536	0.000
-0.02	1.000	1.041	1.283	1.724	2.365	3.207	4.248	5.489	6.931	8.572	0.000
0.00	1.000	1.045	1.291	1.736	2.381	3.227	4.272	5.517	6.963	8.608	0.000
0.02	1.000	1.049	1.299	1.748	2.397	3.247	4.296	5.545	6.995	8.644	0.000
0.04	1.000	1.053	1.307	1.760	2.413	3.267	4.320	5.573	7.027	8.680	0.000
0.06	1.000	1.057	1.315	1.772	2.429	3.287	4.344	5.601	7.059	8.716	0.000
0.08	1.000	1.061	1.323	1.784	2.445	3.307	4.368	5.629	7.091	8.752	0.000
0.10	1.000	1.065	1.331	1.796	2.461	3.327	4.392	5.657	7.123	8.788	0.000
0.12	1.000	1.069	1.339	1.808	2.477	3.347	4.416	5.685	7.155	8.824	0.000
0.14	1.000	1.073	1.347	1.820	2.493	3.367	4.440	5.713	7.187	8.860	0.000
0.16	1.000	1.077	1.355	1.832	2.509	3.387	4.464	5.741	7.219	8.896	0.000
0.18	1.000	1.081	1.363	1.844	2.525	3.407	4.488	5.769	7.251	8.932	0.000
0.20	1.000	1.085	1.371	1.856	2.541	3.427	4.512	5.797	7.283	8.968	0.000