A New Basis for the Solution of the One-Dimensional Transport Equation

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1. Introduction

- We introduce a new family of functions that satisfies the one-dimensional transport equation.

- We demonstrate that this new family is capable of providing very accurate solutions to the transport equation.
2. The problem and its “exact” solution

• We study a family of solutions $u(t,x)$ of the partial differential equation (PDE)

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - v \frac{\partial u}{\partial x}
\]

(1)

with boundary conditions

\[
u(t,0) = 1
\]

\[
\lim_{x \to \infty} u(t,x) = 0
\]

and initial condition

\[
u(0,x) = 0.
\]

• The coefficients $D$ and $v$ are positive and constant.
• This system has the exact solution (found using method of Laplace transforms):

\[
u(t, x) = \frac{1}{2\sqrt{D\pi}} x \exp \left( \frac{v}{2D} x \right) \times \int_0^t \exp \left( -\frac{v^2 \tau}{4D} - \frac{x^2}{4D\tau} \right) d\tau
\]

• We will use this exact solution for comparison purposes.
example of $u(t, x)$ with $v = D = 1$.

- It is clear that we have here

$$\frac{\partial u}{\partial t} > 0 \quad \text{and} \quad \frac{\partial u}{\partial x} < 0.$$
3. Our new family of solutions

- We have discovered that functions of the form

\[ \hat{u}(t, x) = p + q \exp(k(x + \alpha vt)) \]  \hspace{1cm} (2)

satisfy the PDE (1).

- Here \( \alpha \), \( p \), and \( q \) are constants and

\[ k = \frac{v}{D}(\alpha + 1). \]
• We compute the partial derivatives of (2) and recall the conditions

\[
\frac{\partial u}{\partial t} > 0 \quad \text{and} \quad \frac{\partial u}{\partial x} < 0.
\]

• If we apply these conditions to (2), we see that we require

\[
q < 0 \quad \text{and} \quad -1 < \alpha < 0
\]

or

\[
q > 0 \quad \text{and} \quad \alpha < -1.
\]

• Thus we have constraints on how we may select \( \alpha \) and \( q \).
Graphical representation of the constraints

$q < 0$ and $-1 < \alpha < 0$

or

$q > 0$ and $\alpha < -1$
• Because its exponential nature, a single trial function

\[ \hat{u}(t, x) = p + q \exp(k(x + \alpha vt)) \]

cannot hope to capture the shape of the exact solution.

• So we use a piecewise version of \( \hat{u}(t, x) \) to capture this shape.
4. Computational examples

- In these examples, we choose $v = D = 1$.

- We consider separately curves over which each of $t$ and $x$ is fixed.

- The “breakpoints” for the piecewise curves are chosen so that the exact solution has value $1/2$ at these locations.
In these examples, we select the values of $\alpha$, $p$, and $q$ to solve the “minimax” problem. That is, we find $\mathcal{M}$ defined by

$$
\mathcal{M} = \min_w \max_{x \in [a_x, b_x]} |u(10, x) - \tilde{u}(10, x, w)|
$$

or

$$
\mathcal{M} = \min_w \max_{t \in [a_t, b_t]} |u(t, 10) - \tilde{u}(t, 10, w)|,
$$

where

$$
w = \begin{bmatrix}
\alpha \\
p \\
q
\end{bmatrix},
$$

and such that $\alpha$ and $q$ are subject to the constraints given earlier.
Example 1

\[ t = 10, \ x \in [0, 10.9] \]

\[ \mathbf{w} = \begin{bmatrix} \alpha \\ p \\ q \end{bmatrix} = \begin{bmatrix} -0.6 \\ 1.0 \\ -0.08 \end{bmatrix} \]

\[ \mathcal{M} = 0.0685 \]
Example 2

\[ t = 10, \ x \in [10.9, 25] \]

\[
\mathbf{w} = \begin{bmatrix} \alpha \\ p \\ q \end{bmatrix} = \begin{bmatrix} -1.28 \\ 0 \\ 0.304 \end{bmatrix}
\]

\[ \mathcal{M} = 0.0237 \]
Example 3

$x = 10, \ t \in [0, 9.1]$

\[
\mathbf{w} = \begin{bmatrix}
\alpha \\
\ p \\
\ q
\end{bmatrix} = \begin{bmatrix}
-1.29 \\
-0.02 \\
\qquad 0.34
\end{bmatrix}
\]

\[
\mathcal{M} = 0.0432
\]
Example 4

\[ x = 10, \ x \in [9.1, 25] \]

\[
w = \begin{bmatrix} \alpha \\ p \\ q \end{bmatrix} = \begin{bmatrix} -0.61 \\ 1 \\ -0.09 \end{bmatrix}
\]

\[ M = 0.0100 \]
5. Summary and conclusions

- Herein we introduced a family of functions

\[ \hat{u}(t, x) = p + q \exp \left( \frac{v}{D} (\alpha + 1)(x + \alpha vt) \right) \]

each member of which is a solution of the PDE

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - v \frac{\partial u}{\partial x}. \]

- By carefully choosing values of \( \alpha, \ p, \) and \( q, \) good agreement with the exact solution of the PDE (with appropriate boundary and initial conditions) can be obtained.
6. Present and future work

- Improve “minimax” algorithm to obtain superior accuracy.

- Use combinations of our functions in such a way as to satisfy initial and boundary conditions associated with the PDE.

- Use these functions as basis functions in a finite element approach to numerically solve our PDE.