Isomorphisms between small subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$

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Automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$

As usual, $\triangle$ denotes symmetric difference, i.e.,

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

We write $A =^* B$ if $A \triangle B$ is finite and $A \subseteq^* B$ if $A \setminus B$ is finite. The collection of finite subsets of the set ω of all natural numbers is denoted by $\mathtt{fin}$.

$\mathcal{P}(\omega)/\mathtt{fin}$ is the power set algebra of ω factored by the ideal $\mathtt{fin}$, i.e., two sets $A, B \subseteq \omega$ are identified if $A =^* B$.

If $A \subseteq \omega$ we write $A \triangle \mathtt{fin}$ for the set $\{A \triangle F : F \in \mathtt{fin}\}$, which happens to be an element of $\mathcal{P}(\omega)/\mathtt{fin}$.

Every permutation of ω induces an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. Let us call these automorphisms *very trivial*.

If $A, B \subseteq \omega$ are cofinite and $b : A \to B$ is a bijection, then b induces an automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ by letting

$$\varphi(A \triangle \mathtt{fin}) = b[A] \triangle \mathtt{fin}.$$

These automorphisms all called *trivial*. Let Triv denote the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$.

Theorem (van Douwen)

The very trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ form a normal subgroup of the group of trivial automorphisms.

Proof.

If φ is a trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ that is induced by $b : A \to B$, then let

$$\text{index}(\varphi) = |\omega \setminus B| - |\omega \setminus A| .$$

The index of φ is independent of the choice of A and B.
The map index : Triv $\to \mathbb{Z}$ is a homomorphism and the kernel of the homomorphism consists precisely of the very trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$. □

The *shift* s on $\mathcal{P}(\omega)/\mathtt{fin}$ is the trivial automorphism induced by the map

$$S : \omega \to \omega; n \mapsto n + 1.$$

It is well-known that for every trivial automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$, $s^{-1} \neq \varphi \circ s \circ \varphi^{-1}$.

This is because for every $\varphi \in \mathrm{Triv}$ we have

$$\begin{aligned} \mathrm{index}(\varphi \circ s \circ \varphi^{-1}) &= \mathrm{index}(\varphi) + \mathrm{index}(s) - \mathrm{index}(\varphi) \\ &= \mathrm{index}(s) = 1 \neq -1 = \mathrm{index}(s^{-1}). \end{aligned}$$

Question

Is there an automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $s^{-1} = \varphi \circ s \circ \varphi^{-1}$?

Theorem (Shelah, Steprāns, Veličković)

It is consistent with the usual axioms of set theory that every automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ is trivial.

If we want to find an automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ that conjugates s and s^{-1}, we need to construct a non-trivial automorphism.

There are essentially two constructions of non-trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$.

1. Use CH or something similar and define an automorphism of larger and larger subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$, diagonalizing against all trivial automorphisms. (Various people, including Walter Rudin, Shelah, Steprāns, Koppelberg, G.)

2. Start with a suitable model of set theory and force a non-trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ without adding new reals. (Veličković, Steprāns)

A subalgebra A of a Boolean algebra B is a *σ-subalgebra* if for all $b \in B$ the ideal $\{a \in A : a \leq b\}$ of A is countably generated. We write $A \leq_\sigma B$ is A is a σ-subalgebra of B.

Lemma

Let $A \leq_\sigma \mathcal{P}(\omega)/\mathtt{fin}$ be of size $< 2^{\aleph_0}$ and assume that Martin's Axiom holds for countable partial orders. If $x, y \in (\mathcal{P}(\omega)/\mathtt{fin}) \setminus A$, then for every automorphism f of A there is an extension $\bar{f}$ of f to a subalgebra B of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $A \cup \{x\} \subseteq B$ and $\bar{f}(x) \neq y$.

A Boolean algebra A is *tightly σ-filtered* if there is a sequence $(a_\alpha)_{\alpha<|A|}$ of elements of A such that for all $\alpha <|A|$ the algebra A_α generated by $\{a_\beta : \beta < \alpha\}$ is a σ-subalgebra of A.

Theorem

If $\mathcal{P}(\omega)/\mathtt{fin}$ is tightly σ-filtered, then $\mathcal{P}(\omega)/\mathtt{fin}$ has $2^{2^{\aleph_0}}$ automorphisms. In particular, $\mathcal{P}(\omega)/\mathtt{fin}$ has non-trivial automorphisms.

$\mathcal{P}(\omega)/\mathtt{fin}$ is tightly σ-filtered under CH and in models of set theory obtained by adding $\leq \aleph_2$ Cohen-reals to a model of CH.

This method has some limitations.

The combinatorial problems when trying to construct an automorphism φ that conjugates s and s^{-1} seem to be overwhelming. But this might have to do with the fact that we have too much freedom in the construction of such a φ.

Theorem

If $\mathcal{P}(\omega)/\mathtt{fin}$ is tightly σ-filtered, then $2^{\aleph_0} \leq \aleph_2$.

But we would be happy to produce an automorphism that conjugates s and s^{-1} even under CH.

Isomorphisms between countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$

Theorem

Let $f : A \to B$ be an isomorphism between countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. Then there is a very trivial automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $f = \varphi \restriction A$.

Lemma

Let $f : A \to B$ be an isomorphism between countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. Then f extends to an isomorphism $\bar{f} : \bar{A} \to \bar{B}$ between countable atomless subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$.

Theorem (Folklore)

Every countable atomless Boolean algebra is isomorphic to the free Boolean algebra with countably many generators.

Theorem (Folklore)

For every permutation σ on a set X, σ and σ^{-1} are conjugate in the full symmetric group on X.

Corollary

If A is a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$, then there are automorphisms ϕ and ψ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $\psi \restriction A = s \restriction A$ and $\psi^{-1} = \varphi \circ \psi \circ \varphi^{-1}$.

Theorem (Farah)

If A is a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ that is closed under s, then there is and automorphism φ of A such that $s^{-1} \restriction A = \varphi \circ (s \restriction A) \circ \varphi^{-1}$.

The shift and its inverse on small subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$

Definition

A subalgebra A of $\mathcal{P}(\omega)/\mathtt{fin}$ is diagonalized *by $a \in \mathcal{P}(\omega)/\mathtt{fin}$ if for all $b \in A$ either $a \leq b$ or $a \leq \neg b$. A is* shiftclosed *if A is closed under s and s^{-1}.*

Lemma

Let $\bar{x} = (x_n)_{n\in\omega}$ be a strictly increasing sequence in ω and let $a = \{x_n : n \in \omega\} \triangle \mathtt{fin}$. Then

$$D(a) = \{b \in \mathcal{P}(\omega)/\mathtt{fin} : \forall n \in \mathbb{Z}(a \leq s^n(b) \vee a \leq s^n(\neg b))\}$$

is a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$, the largest shiftclosed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ that is diagonalized by a.

Lemma

Let a and $\bar{x}$ be as in the previous lemma. Let $b : \omega \to \omega$ be the permutation that reflects each interval $[x_n, x_{n+1})$ about its median.
Then $D(a)$ is closed under the automorphism φ of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by b and moreover,

$$s^{-1} \restriction D(a) = \varphi \circ (s \restriction D(a)) \circ \varphi^{-1}.$$

How do we find sequences that diagonalize a given subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$?

Lemma (Folklore)

Let $A \subseteq \mathcal{P}(\omega)/\mathtt{fin}$ be a family of size $< 2^{\aleph_0}$ that has the finite intersection property. Then Martin's Axiom implies that there is $a \in \mathcal{P}(\omega)/\mathtt{fin}$ such that $a \leq b$ for all $b \in A$.

Example

Assume MA and let A be a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ of size $< 2^{\aleph_0}$. Let U be an ultrafilter of A. Then there is $a \in \mathcal{P}(\omega)/\mathtt{fin}$ such that $a \leq b$ for all $b \in U$. It is easily checked that a diagonalizes A.

Corollary

Assume MA. Then every set $T \subseteq \mathcal{P}(\omega)/\mathtt{fin}$ of size $< 2^{\aleph_0}$ is contained in a shiftclosed subalgebra A of $\mathcal{P}(\omega)/\mathtt{fin}$ such that on A the shift is conjugate to its inverse.

Proof.

Let B be the subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ generated by T. Then T is of size $< 2^{\aleph_0}$. By MA, B is diagonalized by some $a \in \mathcal{P}(\omega)/\mathtt{fin}$. Now $A = D(a)$ is shiftclosed and on A the shift is conjugate to its inverse. $\square$