Reasonable ultrafilters

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Reasonability

Reasonable ultrafilters on uncountable cardinals were introduced in Shelah [Sh:830][1] in order to suggest a line of research that would in some sense repeat the beautiful theory created around the notion of *P–points on* ω. The definition of reasonable ultrafilters involves two conditions.

- The first demand, so called the *weak reasonability* of an ultrafilter, is a way to guarantee that we are not entering the realm of large cardinals: the considered ultrafilter is required to be *very non-normal*.
- The second part of the definition is a creative re-interpretation of the property that *any countable family of members of the ultrafilter has a pseudo-intersection in the ultrafilter.*

[1] *The combinatorics of reasonable ultrafilters.* **Fund Math** 192 (2006)

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Weak reasonability

Definition: *Let D be a uniform ultrafilter on a regular uncountable cardinal λ. We say that D is weakly reasonable, if for every increasing continuous sequence $\langle \delta_\xi : \xi < \lambda \rangle \subseteq \lambda$ there is a club C of λ such that*

$$\bigcup \{ [\delta_\xi, \delta_{\xi+1}) : \xi \in C \} \notin D.$$

Ultrafilters from filters on small sets

Let $\mathbb{Q}^0_\lambda$ be the collection of all sequences $r = \langle(\alpha_\xi, d_\xi) : \xi < \lambda\rangle$ such that

- $\langle\alpha_\xi : \xi < \lambda\rangle$ is an increasing continuous sequence of ordinals below λ and
- d_ξ is an ultrafilter on the interval $[\alpha_\xi, \alpha_{\xi+1})$.

For $r \in \mathbb{Q}^0_\lambda$ let $\mathrm{fil}(r)$ be the family of subsets of λ which are *eventually large in every interval* $[\alpha_\xi, \alpha_{\xi+1})$, that is

$$\mathrm{fil}(r) = \big\{A \subseteq \lambda : (\exists \zeta < \lambda)(\forall \xi > \zeta)(A \cap [\alpha_\xi, \alpha_{\xi+1}) \in d_\xi)\big\}.$$

(The set $\mathrm{fil}(r)$ is a filter on λ.)
We say that $r \leq^0 s$ if and only if $\mathrm{fil}(r) \subseteq \mathrm{fil}(s)$.
($\leq^0$ is a quasi order on $\mathbb{Q}^0_\lambda$.)

The demand generalizing P-pointness for an ultrafilter D on λ is:

(*) *there is a* $(<\lambda^+)$*–directed (with respect to* $\leq^0$*) family* H *such that* $D = \bigcup\{\mathrm{fil}(r) : r \in H\}$.

The family H as above may be called *a generating family for the ultrafilter* D.

Reasonable ultrafilters are ultrafilters which are weakly reasonable and satisfy the condition (*). (So, reasonable ultrafilters are weakly reasonable ultrafiters with $(<\lambda^+)$–directed generating families.)

The two components are connected

Proposition: [S.Shelah and AR]
Suppose that $\kappa \leq \lambda$ and $H \subseteq \mathbb{Q}^0_\lambda$ is a $(<\kappa)$–directed family such that $D := \bigcup\{\mathrm{fil}(r) : r \in H\}$ is an ultrafilter on λ. If D is not weakly reasonable, then for some club C of λ the quotient ultrafilter D/C is $(<\kappa)$–complete and it contains all clubs of λ.

There may be reasonable ultrafilters

Theorem: [S.Shelah and AR]

- Assume $\lambda = \lambda^{<\lambda}$ and $\diamondsuit_{S^{\lambda^+}_{\lambda}}$ holds. There exists a sequence $\langle r_\xi : \xi < \lambda^+ \rangle \subseteq \mathbb{Q}^0_\lambda$ such that
  - *(i)* $(\forall \xi < \zeta < \lambda^+)(r_\xi \leq^0 r_\zeta)$, and
  - *(ii)* the family $D = \bigcup\limits_{\xi<\lambda^+} \mathrm{fil}(r_\xi)$ is an ultrafilter on λ (so it is a reasonable ultrafilter on λ).
- The forcing notion $\mathbb{Q}^0_\lambda = (\mathbb{Q}^0_\lambda, \leq^0)$ is $(<\lambda^+)$–complete and $\Vdash_{\mathbb{Q}^0_\lambda}$ " $G_{\mathbb{Q}^0_\lambda}$ is a reasonable family generating an ultrafilter ".

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Could there be no reasonable ultrafilters?

Problem: Is it consistent that there are no reasonable ultrafilters on λ?

Theorem: [S.Shelah and AR]
Assume λ is a strongly inaccessible cardinal. Then there is a forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \text{“ } \lambda \text{ is strongly inaccessible and } 2^{\lambda} = \lambda^{++} \text{ and there is no reasonable ultrafilter on } \lambda \text{ with a generating system of size } < 2^{\lambda} \text{ ”}$$

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Small generating systems

Theorem: [S.Shelah and AR]
Assume that λ is a strongly inaccessible cardinal. Then there is a forcing notion $\mathbb{P}$ such that

$\Vdash_{\mathbb{P}}$ " λ is strongly inaccessible and $2^\lambda = \lambda^{++}$ and there is a $(<\lambda^+)$–directed family $H \subseteq \mathbb{Q}^0_\lambda$ such that $|H| = \lambda^+$ and $\mathrm{fil}(H)$ is an ultrafilter on λ, in particular there is a reasonable ultrafilter on λ with generating system of size $< 2^\lambda$ "

Weak reasonability game

Definition:
Let D be a uniform ultrafilter on λ. We define a game $\Game_D$ between two players, Odd and Even, as follows. A play of $\Game_D$ lasts λ steps and during a play an increasing continuous sequence $\bar{\alpha} = \langle \alpha_i : i < \lambda \rangle \subseteq \lambda$ is constructed. The terms of $\bar{\alpha}$ are chosen successively by the two players so that Even chooses the α_i for even i (including limit stages i where she has no free choice) and Odd chooses α_i for odd i. Even wins the play if and only if $\bigcup\{[\alpha_{2i+1}, \alpha_{2i+2}) : i < \lambda\} \in D$.

Proposition: [S.Shelah]
Assume D is a uniform ultrafilter on λ.

- If D is not weakly reasonable, then Odd has a winning strategy in the game $\Game_D$.
- If λ is strongly inaccessible and Odd has a winning strategy in $\Game_D$, then D is not weakly reasonable.

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Theorem: [S.Shelah and AR]
Assume that there exists a strongly inaccessible cardinal. Then some forcing notion forces that

" there is a $\leq^*$–increasing sequence $\langle r_\xi : \xi < \omega_2 \rangle \subseteq \mathbb{Q}^0_\lambda$ such that $D := \bigcup \big(\{ \mathrm{fil}(r_\xi) : \xi < \omega_2 \} \big)$ is a very reasonable ultrafilter on ω_1 but Odd has a winning strategy in the game $\Game_D$ ".