An initial segment property of mice

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The idea is to use directed systems of mice, like that used by Woodin to calculate $\mathrm{HOD}^{L(\mathbb{R})}$, to better understand determinacy phenomena as well as improve known results. Some hallmarks include:

- (Steel - precursor to Woodin's analysis) All regular cardinals of $L(\mathbb{R})$ are measurable, $\mathrm{HOD}^{L(\mathbb{R})}|\Theta$ is a fine structural model, as a consequence GCH holds in $\mathrm{HOD}^{L(\mathbb{R})}$ (as well as $\diamondsuit_\kappa$ and $\square_\kappa$ where possible).
- (Neeman) Analyzed the supercompactness measures on $\mathcal{P}_{\omega_1}(\lambda)$ and reanalyzed some of the supercompactness measures $\mathcal{P}_\kappa(\lambda)$ for κ regular Suslin cardinals other than ω_1.
- (Neeman) The Kechris-Martin result on closure of Π^1_3 under $< \delta^1_3$-length unions.
- (Woodin) Used his analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ to produce an entirely new hierarchy of mice based on $\mathrm{HOD}^{L(\mathbb{R})}$ which he uses to analyze AD^+ and the Ω-conjecture.

One goal for the project is to understand the coding lemma and some of its strengthenings.

A second goal is to understand the partition properties and their generalizations.

Jackson's analysis of measures shows that every bounded subset of ω_ω has a code which is $\Game_{<\omega^2}$-Π^1_1 and hence all subsets of ω_ω are coded in $\bigcup_\omega \Game_{<\omega^2}$-$\Pi^1_1$ which is well below Δ^1_3. This is better than what the standard coding lemma or even Kunen's coding of subsets of ω_ω gives.

A goal was to look at how subsets of ω_ω are coded by mice (a consequence of Woodin's HOD analysis) and then use this to reprove Jackson's result by analyzing the complexity of the directed systems involved.

Once accomplished this technique should not break down at the same place where Jackson's analysis runs out of steam and we should have the correct coding lemma for subsets of arbitrary Suslin cardinals of cofinality ω.

Jackson's analysis of measures also shows that the strong partition property holds at every regular Suslin cardinal that he can analyze.

If the directed system approach can be made to work just at $\delta^1_3 = \omega_{\omega+1}$, then the feeling is it should work at all regular Suslin cardinals.

Steel noticed that if one could prove that Π^1_3 is closed under quantification by the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_\omega)$, then the directed system could be used to reprove $\delta^1_3 \rightarrow (\delta^1_3)^{\delta^1_3}$

Using standard determinacy arguments Jackson unfortunately proved

Theorem 1 (Jackson). Π^1_3 is not closed under quantification by the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_\omega)$.

In a "glass half full" move Jackson's negative result lead to a nice question that is still open: Is Π^1_3 is closed under quantification by the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_n)$ for each n?

Toward a partial answer, Jackson applied his analysis of measures showing the answer is yes for $n = 1, 2$, it is open for all other n. - This is currently one of the goals we are pursuing.

The next result arose not from the question of how to code subsets of ω_ω, but instead was simply an attempt to analyze the coding of ordinals $< \omega_2$.

Under AD the club filter on ω_1 is an ultrafilter, call it μ, and $\prod \omega_1/\mu = \omega_2$ so an obvious set of codes for ordinals $< \omega_2$ are just the functions $f : \omega_1 \to \omega_1$.

Say that a transitive M has the *initial segment property* for ω_2 if for all $f, g : \omega_1 \to \omega_1$ with $f \in M$ and $g <_\mu f$, there is $g^* \in M$ with $g^* =_\mu g$. So that if $\mu \cap M \in M$, then M correctly computes $[f]_\mu$.

Woodin has shown that not all mice have the initial segment property and in fact a counterexample can be found in the form $\mathrm{HOD}^{L(\mathbb{R})}[g]$ for some $g : \omega_1 \to \omega_1$ such that $\mu \cap \mathrm{HOD}^{L(\mathbb{R})}[g] \in \mathrm{HOD}^{L(\mathbb{R})}[g]$. (This is in some sense very close to the models which we show do have the initial segment property.)

Essentially we show that any mouse M that is the limit of a directed system of mice, like that used in the HOD-calculation, has the initial segment property.

In an attempt to axiomatize this call a mouse M *reasonable* if:

- ω_1^V is the critical point of the least total extender E_0 on M's sequence and $E_0 = \mu \cap M$.
- M and $j_\mu(M)$ have a successful comparison in $L(\mathbb{R})$. (This is the tricky part.)
- M is φ-*minimal* in the sense that $M \models \varphi$, but for all $\alpha < \mathrm{dom}(E^M)$, $L[E|\alpha] \models \neg\varphi$ (or some slight (technical) variant of this.)

$\mathrm{HOD}^{L(\mathbb{R})}$ is reasonable.

Theorem. Reasonable models have the initial segment property at ω_2.

Obvious questions include:

- Does this help understand the coding of subsets of ω_2?
- What happens at ω_3 and beyond?
- Does this give a new proof that Π^1_3 is closed under the super compactness measure on $\mathcal{P}_{\omega_1}(\omega_2)$?