Solution to a Problem of Van Douwen

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Outline

Basic Definitions

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Two functions f and g in ω^ω are said to be almost disjoint (a.d.) if they agree only at finitely many places – i.e. if $f \cap g$ is finite.

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An a.d. family $\mathscr{A} \subset \omega^\omega$ is MAD if for every $f \in \omega^\omega$, there is $h \in \mathscr{A}$ such that $h \cap f$ is infinite. Sometimes such a family is called Maximal Eventually Different.

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An a.d. family $\mathscr{A} \subset [X]^\omega$ is MAD if every $b \in [X]^\omega$ infinitely hits some $a \in \mathscr{A}$.

Note that we are allowing finite families to be MAD.

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We will call such a family a Van Douwen MAD family.

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Last August I proved:

Theorem ([2])

There is a Van Douwen MAD family of size $\mathfrak{c}$.

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- Build an object of size $\mathfrak{c}$ in $\mathfrak{c}$ steps.
- You enumerate all your "requirements" in $\mathfrak{c}$ steps. In this case, you let $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$ all infinite partial functions.
- You construct $\mathscr{A} = \{h_\alpha : \alpha < \mathfrak{c}\}$ in $\mathfrak{c}$ steps and take care of the "α-th requirement" at stage α. In this case, that means if f_α is a.d. from $\{h_\beta : \beta < \alpha\}$, you make sure that $h_\alpha \cap f_\alpha$ is infinite.

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- In general, you need assumptions that go beyond ZFC to carry this out – too many "requirements".
- At stage α, you want to find $h \in \omega^\omega$ which hits f_α and is a.d. from $\{h_\beta : \beta < \alpha\}$.
- But may be there are no total functions a.d. from $\{h_\beta : \beta < \alpha\}$ (remember f_α is only a partial function).

Rephrase the Problem . . .

Definition

Let $\mathscr{A} \subset \omega^{\omega}$ *be a.d. and let* $g \in \omega^{\omega}$*. Define* $\mathscr{A} \cap g$ *to be* $\{h \cap g : h \in \mathscr{A} \wedge |h \cap g| = \omega\}$*.*

Note that this is an a.d. family of subsets of the countable set g.

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Does there exist an a.d. family $\mathscr{A} \subset \omega^\omega$ such that $\mathrm{tr}\left(\mathscr{A}\right) = \omega^\omega$?

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There is a family $\mathcal{F} = \{f_\alpha : \alpha < \mathrm{non}(\mathcal{M})\} \subset \omega^\omega$ *such that no infinite partial function is a.d. from it.*

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Fact

There is a family $\mathcal{F} = \{f_\alpha : \alpha < \text{non}(\mathcal{M})\} \subset \omega^\omega$ *such that no infinite partial function is a.d. from it.*

Van Douwen's Problem

Does there exist an a.d. family $\mathscr{A} \subset \omega^\omega$ *such that* $\mathcal{F} \subset \text{tr}(\mathscr{A})$*?*

Getting f_0 into $\mathrm{tr}\left(\mathscr{A}_0\right)$

- You build $\mathscr{A} = \bigcup \mathscr{A}_\alpha$ in $\mathrm{non}\left(\mathcal{M}\right)$ steps. At stage α you ensure $f_\alpha \in \mathrm{tr}\left(\mathscr{A}_\alpha\right)$

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- Get $\mathscr{A}_0$ by gluing together pieces of f_0 with pieces of other functions.
- Choose a MAD family $\{a_\xi : \xi < \mathfrak{c}\} \subset [\omega]^\omega$. The pieces of f_0 are $\{f_0 \upharpoonright a_\xi\}$.
- Ensure that $\mathscr{A}_0 = \{h_\xi : \xi < \mathfrak{c}\} \subset \omega^\omega$, where $f_0 \upharpoonright a_\xi \subset h_\xi$
- To ensure $\mathscr{A}_0$ is a.d. need an a.d. family $\mathscr{C}_0 \subset \omega^\omega$ which is a.d. from f_0.

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- You build $\mathscr{A} = \bigcup \mathscr{A}_\alpha$ in $\mathrm{non}(\mathcal{M})$ steps. At stage α you ensure $f_\alpha \in \mathrm{tr}(\mathscr{A}_\alpha)$
- Get $\mathscr{A}_0$ by gluing together pieces of f_0 with pieces of other functions.
- Choose a MAD family $\{a_\xi : \xi < \mathfrak{c}\} \subset [\omega]^\omega$. The pieces of f_0 are $\{f_0 \restriction a_\xi\}$.
- Ensure that $\mathscr{A}_0 = \{h_\xi : \xi < \mathfrak{c}\} \subset \omega^\omega$, where $f_0 \restriction a_\xi \subset h_\xi$
- To ensure $\mathscr{A}_0$ is a.d. need an a.d. family $\mathscr{C}_0 \subset \omega^\omega$ which is a.d. from f_0.
- Can continue this because of some combinatorial properties of $\mathrm{non}(\mathcal{M})$.

Open Question

Definition

Let $\mathfrak{a}_\mathfrak{e}$ be the least size of a MAD family in ω^ω and let $\mathfrak{a}_\mathfrak{v}$ be the least size of a Van Douwen MAD family in ω^ω.

Question

Is it consistent to have $\mathfrak{a}_\mathfrak{e} < \mathfrak{a}_\mathfrak{v}$?

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