On Hurewicz subsets of $\mathbb{R}^{\mathbb{N}}$

Marion Scheepers

BEST 17

Outline

Outline

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

$L \subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

$ZFC + CH \vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

$S \subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

$ZFC + CH \vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

$ZFC + CH \vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

$ZFC + CH \vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH ⊢ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH ⊢ There is a Sierpiński set.

Lusin and Sierpiński sets.

Lusin sets (1913/4)

Definition

L $\subset \mathbb{R}$ is a Lusin set if:

- $|L| = 2^{\aleph_0}$ and
- for each nowhere dense set $N \subset \mathbb{R}$, $|L \bigcap N| \leq \aleph_0$

Theorem (Lusin-Mahlo)

ZFC + CH $\vdash$ There is a Lusin set.

Sierpiński sets (1924)

Definition

S $\subset \mathbb{R}$ is a Sierpiński set if:

- $|S| = 2^{\aleph_0}$ and
- for each measure zero set $N \subset \mathbb{R}$, $|S \bigcap N| \leq \aleph_0$

Theorem (Sierpiński)

ZFC + CH $\vdash$ There is a Sierpiński set.

Three fundamental results.

Theorem (Rothberger)

ZFC $\vdash$ CH $\Leftrightarrow$ there are both a Lusin set and a Sierpiński set.

Theorem (Galvin-Mycielski-Solovay)

If L is a Lusin set then for each first category set M, L+M$\neq \mathbb{R}$.

Theorem (Pawlikowski)

If L is a Sierpiński set then for each Lebesgue measure zero set N, L+N$\neq \mathbb{R}$.

Three fundamental results.

Theorem (Rothberger)

ZFC $\vdash$ *CH* $\Leftrightarrow$ *there are both a Lusin set and a Sierpiński set.*

Theorem (Galvin-Mycielski-Solovay)

If L is a Lusin set then for each first category set M, L+M$\neq \mathbb{R}$.

Theorem (Pawlikowski)

If L is a Sierpiński set then for each Lebesgue measure zero set N, L+N$\neq \mathbb{R}$.

Three fundamental results.

Theorem (Rothberger)

ZFC $\vdash$ CH $\Leftrightarrow$ there are both a Lusin set and a Sierpiński set.

Theorem (Galvin-Mycielski-Solovay)

If L is a Lusin set then for each first category set M, L+M$\neq \mathbb{R}$.

Theorem (Pawlikowski)

If L is a Sierpiński set then for each Lebesgue measure zero set N, L+N$\neq \mathbb{R}$.

Three fundamental results.

Theorem (Rothberger)

ZFC $\vdash$ CH $\Leftrightarrow$ there are both a Lusin set and a Sierpiński set.

Theorem (Galvin-Mycielski-Solovay)

If L is a Lusin set then for each first category set M, L+M$\neq \mathbb{R}$.

Theorem (Pawlikowski)

If L is a Sierpiński set then for each Lebesgue measure zero set N, L+N$\neq \mathbb{R}$.

Outline

Notions of countable dimensionality.

$C \subset \mathbb{R}^N$ is

countable dimensional if:

C is a union of countably many finite dimensional sets (1928).

strongly countable dimensional if:

C is a union of countably many closed finite dimensional sets (1959/60).

Notions of countable dimensionality.

$C \subset \mathbb{R}^N$ is

countable dimensional if:

C is a union of countably many finite dimensional sets (1928).

strongly countable dimensional if:

C is a union of countably many closed finite dimensional sets (1959/60).

Notions of countable dimensionality.

$C \subset \mathbb{R}^N$ is

countable dimensional if:

C is a union of countably many finite dimensional sets (1928).

strongly countable dimensional if:

C is a union of countably many closed finite dimensional sets (1959/60).

Notions of countable dimensionality.

$C \subset \mathbb{R}^N$ is

countable dimensional if:

C is a union of countably many finite dimensional sets (1928).

strongly countable dimensional if:

C is a union of countably many closed finite dimensional sets (1959/60).

Notions of countable dimensionality.

$C \subset \mathbb{R}^{\mathbb{N}}$ is

countable dimensional if:

C is a union of countably many finite dimensional sets (1928).

strongly countable dimensional if:

C is a union of countably many closed finite dimensional sets (1959/60).

Basic Facts.

Theorem (Hurewicz)

$\mathbb{R}^{\mathbb{N}}$ is not countable dimensional.

Theorem (Nagami-Smirnov)

$\mathbb{R}^{\mathbb{N}}$ is a union of $\aleph_1$ countable dimensional sets.

Theorem (Folklore)

There are countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ which are not strongly countable dimensional.

Basic Facts.

Theorem (Hurewicz)

$\mathbb{R}^{\mathbb{N}}$ is not countable dimensional.

Theorem (Nagami-Smirnov)

$\mathbb{R}^{\mathbb{N}}$ is a union of $\aleph_1$ countable dimensional sets.

Theorem (Folklore)

There are countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ which are not strongly countable dimensional.

Basic Facts.

Theorem (Hurewicz)

$\mathbb{R}^{\mathbb{N}}$ *is not countable dimensional.*

Theorem (Nagami-Smirnov)

$\mathbb{R}^{\mathbb{N}}$ *is a union of* $\aleph_1$ *countable dimensional sets.*

Theorem (Folklore)

There are countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ *which are not strongly countable dimensional.*

Basic Facts.

Theorem (Hurewicz)

$\mathbb{R}^{\mathbb{N}}$ is not countable dimensional.

Theorem (Nagami-Smirnov)

$\mathbb{R}^{\mathbb{N}}$ is a union of $\aleph_1$ countable dimensional sets.

Theorem (Folklore)

There are countable dimensional subsets of $\mathbb{R}^{\mathbb{N}}$ which are not strongly countable dimensional.

Outline

Hurewicz sets (1932)

Definition

A subset H or $\mathbb{R}^{\mathbb{N}}$ is a Hurewicz set if

- $|H| = 2^{\aleph_0}$ and
- for each countable dimensional set $N \subset \mathbb{R}^{\mathbb{N}}$, $|H \bigcap N| \leq \aleph_0$

Hurewicz sets (1932)

Definition

A subset H or $\mathbb{R}^{\mathbb{N}}$ is a Hurewicz set if

- $|H| = 2^{\aleph_0}$ and
- for each countable dimensional set $N \subset \mathbb{R}^{\mathbb{N}}$, $|H \bigcap N| \leq \aleph_0$

Hurewicz sets (1932)

Definition

A subset H or $\mathbb{R}^{\mathbb{N}}$ is a Hurewicz set if

- $|H| = 2^{\aleph_0}$ and
- for each countable dimensional set $N \subset \mathbb{R}^{\mathbb{N}}$, $|H \bigcap N| \leq \aleph_0$

Two Theorems and a Conjecture

Theorem (Hurewicz)

ZFC $\vdash$ CH $\Leftrightarrow$ There is a Hurewicz set in $\mathbb{R}^{\mathbb{N}}$.

Theorem

If $H \subset \mathbb{R}^{\mathbb{N}}$ is a Hurewicz set, then $H + N \neq \mathbb{R}^{\mathbb{N}}$ (indeed, is first category) for each strongly countable dimensional set $N \subset \mathbb{R}^{\mathbb{N}}$.

Conjecture

If $H \subset \mathbb{R}^{\mathbb{N}}$ is a Hurewicz set, then $H + N \neq \mathbb{R}^{\mathbb{N}}$ for all countable dimensional sets $N \subset \mathbb{R}^{\mathbb{N}}$.

Two Theorems and a Conjecture

Theorem (Hurewicz)

ZFC $\vdash$ *CH* $\Leftrightarrow$ *There is a Hurewicz set in* $\mathbb{R}^{\mathbb{N}}$.

Theorem

If $H \subset \mathbb{R}^{\mathbb{N}}$ *is a Hurewicz set, then* $H + N \neq \mathbb{R}^{\mathbb{N}}$ *(indeed, is first category) for each strongly countable dimensional set* $N \subset \mathbb{R}^{\mathbb{N}}$.

Conjecture

If $H \subset \mathbb{R}^{\mathbb{N}}$ is a Hurewicz set, then $H + N \neq \mathbb{R}^{\mathbb{N}}$ for all countable dimensional sets $N \subset \mathbb{R}^{\mathbb{N}}$.

Two Theorems and a Conjecture

Theorem (Hurewicz)

ZFC $\vdash$ *CH* $\Leftrightarrow$ *There is a Hurewicz set in* $\mathbb{R}^{\mathbb{N}}$.

Theorem

If $H \subset \mathbb{R}^{\mathbb{N}}$ *is a Hurewicz set, then* $H + N \neq \mathbb{R}^{\mathbb{N}}$ *(indeed, is first category) for each strongly countable dimensional set* $N \subset \mathbb{R}^{\mathbb{N}}$.

Conjecture

If $H \subset \mathbb{R}^{\mathbb{N}}$ is a Hurewicz set, then $H + N \neq \mathbb{R}^{\mathbb{N}}$ for all countable dimensional sets $N \subset \mathbb{R}^{\mathbb{N}}$.

Two Theorems and a Conjecture

Theorem (Hurewicz)

ZFC $\vdash$ *CH* $\Leftrightarrow$ *There is a Hurewicz set in* $\mathbb{R}^{\mathbb{N}}$*.*

Theorem

If $H \subset \mathbb{R}^{\mathbb{N}}$ *is a Hurewicz set, then* $H + N \neq \mathbb{R}^{\mathbb{N}}$ *(indeed, is first category) for each strongly countable dimensional set* $N \subset \mathbb{R}^{\mathbb{N}}$*.*

Conjecture

If $H \subset \mathbb{R}^{\mathbb{N}}$ is a Hurewicz set, then $H + N \neq \mathbb{R}^{\mathbb{N}}$ for all countable dimensional sets $N \subset \mathbb{R}^{\mathbb{N}}$.