Is $\square(\omega+1)^\omega$ paracompact?

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1. Preliminaries

$\square_{i\in I}X_i$: base is all $\Pi_{i\in I}u_i$, each u_i open

Very Old Results

1. $\Box_{n<\omega} X_n$ is paracompact if

(Williams) $\mathfrak{d} = \omega_1$ and each X_n compact with weight $\leq \omega_1$

(van Douwen) $\mathfrak{b} = d$ and each X_n compact metrizable

(Roitman) $\mathfrak{d} = \mathfrak{c}$ or add cofinally many Cohen reals and each X_n compact first countable.

2. (Lawrence) $\Box(\omega+1)^{\omega_1}$ not normal.

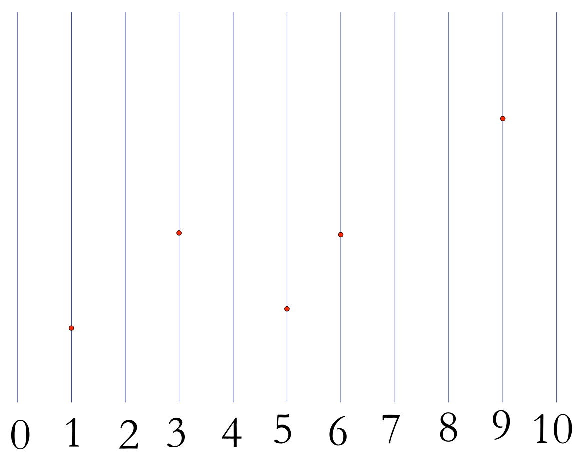
3. (Kunen, van Douwen) $X \times \Box(\omega+1)^{\omega}$ need not be normal if $\chi(X)$ or $|X|$ large or X not the right kind of compact.

Hence focus on $\Box(\omega+1)^{\omega}$.

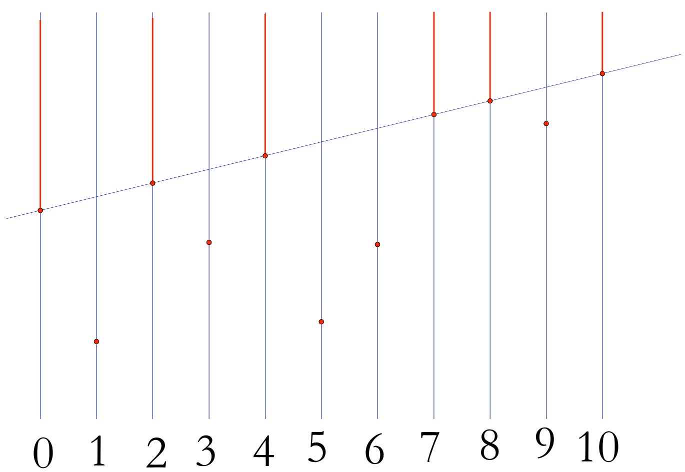
$\nabla(\omega+1)^\omega$ is $\Box(\omega+1)^\omega/_{=^*}$

$P = \{x\text{: } x$ a partial function from ω to ω with dom x infinite co-infinite$\}$

A typical function in P:



Neighborhoods $N(x, f) = \{y \in P : y \supset^* x$ and $(y \setminus x) >^* f\}$.



P not even T_0, but fine transversals (= transversal with respect to $=^*$) are T_2.

Theorem: $\Box(\omega+1)^\omega$ is paracompact iff $\nabla(\omega+1)^\omega$ is ultraparacompact (upc) iff every fine transversal of P is ultraparacompact (= every open cover has a pairwise disjoint covering refinement).

So we search for subsets X of P whose fine transversals are upc subspaces of P (= every open cover of P has a pairwise disjoint refinement covering X — stronger than "upc + subspace").

2. Strongly discrete subspaces

Y strongly discrete iff it has a discrete open cover $\{u_y : y \in Y\}$.

Theorem. If, for each $\alpha < \mathfrak{b}$, each P_α is closed strongly discrete in $P \setminus \bigcup_{\beta<\alpha} P_\beta$, then $\bigcup_{\alpha<\mathfrak{b}} P_\alpha$ is ultraparacompact.

Theorem. Any fine transversal of the following subspaces of P is closed strongly discrete:

$B = \{x \in P : x$ is bounded below some constant function$\}$

$Q_f = \{x \in P : x \leq^* f\}$

$Q_z = \{x \in P : x \subseteq^* z\}$.

These two theorems already give us a lot of upc subspaces of P.

3. $\approx_\perp$

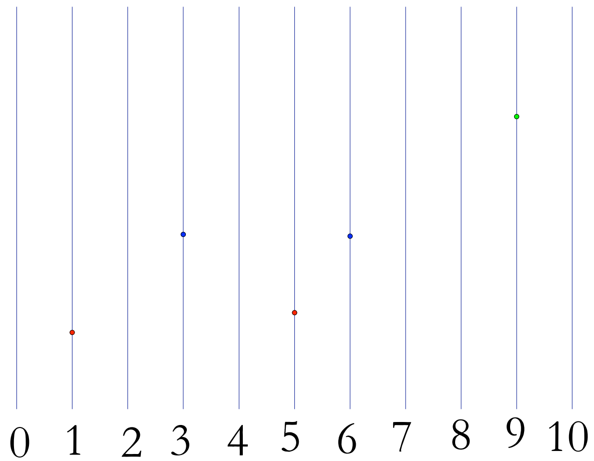
The main tool is equivalence relations coarser than $=^*$. The search is for equivalence relations $\approx$ so that an $\approx$-transversal is a upc subspace.

The first one is $\approx_\perp$.

$x_\perp = \{(n, x(n)) : \forall m > n \text{ if } m \in \text{dom } x \text{ then } x(n) \leq x(m)\}.$

$x_0 = x_\perp$

$x_{n+1} = (x \setminus \bigcup_{m \leq n} x_m)_\perp$



$x \approx_{\perp} y$ iff $\forall n \; x_n =^* y_n$

$P_n = \{x : x_{n+1}$ is finite$\}$

$P_\omega = P \setminus \bigcup_{n<\omega} P_n$.

Fact A $\approx_\perp$-transversal of P_n is closed strongly separated in $P \setminus \bigcup_{m<n} P_m$.

Fact A $\approx_\perp$-transversal of P_ω is closed strongly separated in $P \setminus \bigcup_{m<\omega} P_m$.

Corollary A $\approx_\perp$-transversal of P is a upc subspace of P.

4. $\approx_{\vec{k}}$

$\vec{k} = \{k_\alpha : \alpha < \mathfrak{b}\}$ is an

unbounded

increasing mod finite

family of increasing functions in ω^ω

Fix $x \in P$.

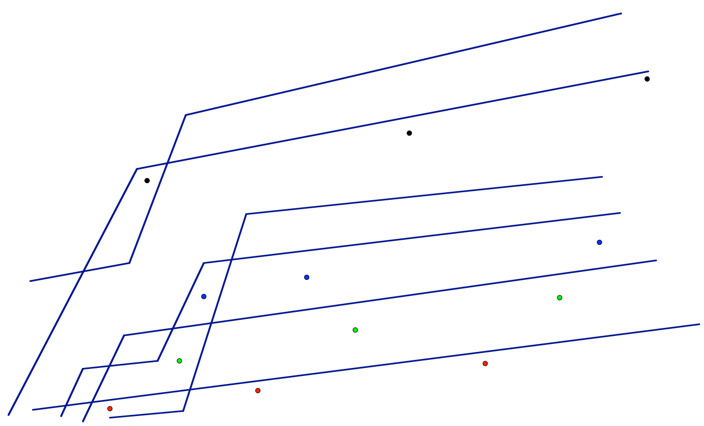
$$\beta_{x,0} = \text{least } \beta \text{ with } x \not>^* k_\beta$$

$$x_{\beta_{x,0}} = \{(n, x(n)) : x(n) < k_{\beta_{x,0}}(n)\}.$$

For $\alpha < \omega_1$,

$$\beta_{x,\alpha} = \text{least } \beta \text{ with } (x \setminus \bigcup_{\gamma<\alpha} x_{\beta_{x,\gamma}} \not>^* k_\beta.$$

$$x_{\beta_{x,\alpha}} = \{(n, x(n)) : x(n) < k_{\beta_{x,\alpha}}(n)\}.$$



$E_x = \{\beta_{x,\alpha} : \alpha < \omega_1\}$

For $\beta = \beta_{x,\alpha}$ we write x_β instead of $x_{\beta_{x,\alpha}}$.

$x \approx_{\vec{k}} y$ iff

$E_x = E_y$

$\forall \beta \in E_x \; x_\beta =^* y_\beta$.

$P_\alpha = \{x : E_x \text{ has order type } \alpha\}$.

Fact A $\approx_{\vec{k}}$-transversal of P_α is closed strongly separated in $P \setminus \bigcup_{\beta<\alpha} P_\beta$.

Corollary A $\approx_{\vec{k}}$-transversal of P is a upc subspace of P.

5. Getting finer equivalence relations

Suppose $\approx$ has some sufficiently nice properties (not defined here)

[these properties make a $\approx$-transversal a upc subspace]

We refine $\approx$ as follows:

Let Y be a $\approx$-transversal.

For $y \in Y$ define $\approx_y$ on $\{x : x \approx y\}$:

$x \approx_y z$ iff

$$x \setminus y \approx z \setminus y$$

$$x \cap y =^* z \cap y$$

$\approx_y = \bigcup_{y \in Y} \approx_y$.

Fact $Y \cup Y_1$ is a upc subspace of P if Y_1 is a $\approx_Y$-transversal.

Here is the notation we need to iterate:

$\approx = \approx_0; Y = Y_0$

$\approx_{\alpha+1} = (\approx_\alpha)_{Y_\alpha}$; where Y_α is a $\approx_\alpha$-transversal.

At limits, take intersections: $x \approx_\alpha y$ iff

$\forall \beta < \alpha \; x \approx_\beta y$

We write $\approx_\alpha$ for this equivalence relation.
tion.

Theorem If $\approx$ is sufficiently nice, and, for each $\alpha < \mathfrak{b}, \approx_\alpha, Y_\alpha$ as above, then $\bigcup_{\alpha<\mathfrak{b}} Y_\alpha$ is a upc subspace of P.