Partitions and Near Coherence

Andreas Blass
Mathematics Department
University of Michigan
Ann Arbor, MI 48109–1043, U.S.A.
ablass@umich.edu

Preliminaries

A non-principal ultrafilter $\mathcal{U}$ on ω is *selective* if every function on ω becomes one-to-one or constant when restricted to a suitable set in $\mathcal{U}$.

$\mathcal{U}$ is a *P-point* if every function on ω becomes finite-to-one or constant when restricted to a suitable set in $\mathcal{U}$.

$f(\mathcal{U}) = \{X : f^{-1}(X) \in \mathcal{U}\}$. This is the ultrafilter generated by $\{f(A) : A \in \mathcal{U}\}$. If f is a bijection, then $\mathcal{U}$ and $f(\mathcal{U})$ are *isomorphic*.

$\mathcal{U}$ and $\mathcal{V}$ on ω are *nearly coherent* if there is a finite-to-one $f : \omega \to \omega$ with $f(\mathcal{U}) = f(\mathcal{V})$. In this case, there also exists a monotone such f.

Two selective ultrafilters are nearly coherent if and only if they are isomorphic. More generally, an ultrafilter $\mathcal{U}$ is nearly coherent with a selective $\mathcal{V}$ if and only if $f(\mathcal{U}) = \mathcal{V}$ for some finite-to-one (monotone) f.

$[X]^k$ is the set of all subsets of X of cardinality k. Here $k \leq \omega$.

Partition Theorems

Ramsey: If $[\omega]^2$ is partitioned into finitely many pieces, then there is an infinite $H \subseteq \omega$ with $[H]^2$ included in one piece.

Kunen: If $\mathcal{U}$ is a selective ultrafilter, then the H in Ramsey's theorem can be taken to be $\in \mathcal{U}$.

Silver: If $[\omega]^\omega$ is partitioned into an analytic piece and a coanalytic piece, then there is an infinite $H \subseteq \omega$ with $[H]^\omega$ included in one piece.

Mathias: If $\mathcal{U}$ is a selective ultrafilter, then the H in Silver's theorem can be taken to be $\in \mathcal{U}$.

Homogeneity for Two Disjoint Sets

Trivial consequence of Ramsey's Theorem:

If $[\omega]^2$ is partitioned into finitely many pieces, then there are disjoint, infinite sets $A, B \subseteq \omega$ such that all the pairs $\{a, b\}$ with $a \in A$ and $b \in B$ lie in the same piece.

Is there an ultrafilter version of this?

Obviously not with A and B in the same ultrafilter.

Not with different ultrafilters either.

Given $\mathcal{U} \neq \mathcal{V}$, fix an $X \in \mathcal{U} - \mathcal{V}$. Partition $[\omega]^2$ so that pairs $\{m < n\}$ with $m \in X$ and $n \notin X$ are in a different piece from those with $m \notin X$ and $n \in X$. If $A \in \mathcal{U}$ and $B \in \mathcal{V}$, then pairs $\{a, b\}$ with $a \in A$ and $b \in B$ occur in both of these pieces.

Homogeneity from Two Ultrafilters

So the best we can hope for is the following polarized partition relation for a pair of distinct ultrafilters $\mathcal{U}$ and $\mathcal{V}$.

$$\omega \to \begin{bmatrix} \mathcal{U} \\ \mathcal{V} \end{bmatrix}^2_2$$

means that, whenever $[\omega]^2$ is partitioned into finitely many pieces, there are disjoint sets $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that all the pairs $\{a, b\}$ with $a \in A$ and $b \in B$ lie in only two of the pieces.

We can also take the order into account explicitly.

$$\omega \to (\mathcal{U} : \mathcal{V})^2$$

means that, whenever $[\omega]^2$ is partitioned into finitely many pieces, there are disjoint sets $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that all the pairs $\{a, b\}$ with $a \in A$, $b \in B$, and $a < b$ lie in the same piece.

Under what conditions on $\mathcal{U}$ and $\mathcal{V}$ do these partition relations hold?

Infinitary Partition Theorems

Trivial consequence of Silver's Theorem:

If $[\omega]^\omega$ is partitioned into an analytic and a co-analytic piece, then there are disjoint, infinite $A, B \subseteq \omega$ such that one piece of the partition contains all of the infinite sets

$$X = \{x_0 < x_1 < \cdots < x_n < \dots\}$$

with some x_n's in A and the rest in B.

For an ultrafilter version, we must keep track of which x_n's come from which of A and B. Let Z be any subset of ω and define

$$\omega \underset{\text{analytic}}{\overset{Z}{\longrightarrow}} (\mathcal{U}, \mathcal{V})^\omega$$

to mean that, whenever $[\omega]^\omega$ is partitioned into an analytic and a coanalytic piece, then there are disjoint sets $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that one piece contains all the sets X as above with $x_n \in A$ for $n \in Z$ and $x_n \in B$ for $n \in \omega - Z$.

Sufficient Conditions for Partition Theorems

Long ago (1988) I showed that, if $\mathcal{U}$ and $\mathcal{V}$ are non-isomorphic, selective ultrafilters, then $\omega \underset{\text{analytic}}{\overset{Z}{\rightarrow}} (\mathcal{U}, \mathcal{V})^\omega$ holds, for all $Z \subseteq \omega$. It follows that we also have $\omega \rightarrow (\mathcal{U} : \mathcal{V})^2$ and therefore also $\omega \rightarrow \begin{bmatrix} \mathcal{U} \\ \mathcal{V} \end{bmatrix}^2_2$.

But is the hypothesis of selectivity really needed for these partition relations? Sometimes it is, but often it can be weakened, at the cost of strengthening the non-isomorphism hypothesis.

Necessary and Sufficient Conditions for Ultrafilter Partition Theorems

In all of the following, unless the contrary is stated, $\mathcal{U}$ and $\mathcal{V}$ are non-principal ultrafilters on ω.

Theorem 1 *The following are equivalent:*

- $\omega \to \begin{bmatrix}\mathcal{U}\\ \mathcal{V}\end{bmatrix}^2_2$.
- $\omega \underset{analytic}{\overset{Even}{\longrightarrow}} (\mathcal{U}, \mathcal{V})^\omega$.
- *$\mathcal{U}$ and $\mathcal{V}$ are P-points and are not nearly coherent.*

Here "Even" means the set of even integers. Of course the same result holds for Odd. But the story is different for other choices of Z. Also, if we separate $\omega \to \begin{bmatrix}\mathcal{U}\\ \mathcal{V}\end{bmatrix}^2_2$ into its two constituents, $\omega \to (\mathcal{U} : \mathcal{V})^2$ and $\omega \to (\mathcal{V} : \mathcal{U})^2$, the criterion for each constituent is weaker (but more complicated).

Here are the conditions for

$$\omega \to (\mathcal{U} : \mathcal{V})^2$$

in the form of a flowchart.

$$\begin{array}{ccc}
\mathcal{V} \text{ a P-point?} & \stackrel{\text{No}}{\Longrightarrow} & \text{No} \\
\text{Yes}\Downarrow & & \\
\mathcal{U}, \mathcal{V} \text{ nearly coherent?} & \stackrel{\text{No}}{\Longrightarrow} & \text{Yes} \\
\text{Yes}\Downarrow & & \\
\text{Nearly coherent with selective?} & \stackrel{\text{No}}{\Longrightarrow} & \text{No} \\
\text{Yes}\Downarrow & & \\
\text{see next slide} & &
\end{array}$$

Theorem 2 *Assume $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are all nearly coherent, $\mathcal{V}$ is a P-point, and $\mathcal{W}$ is selective. Then there is a finite-to-one, monotone function f with $f(\mathcal{U}) = f(\mathcal{V}) = \mathcal{W}$. The following are equivalent.*

- $\omega \to (\mathcal{U} : \mathcal{V})^2$.
- *There are sets $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that, whenever $a \in A$, $b \in B$, and $f(a) = f(b)$, then $b < a$.*
- *For every function $g : \omega \to \omega$, there is $A \in \mathcal{U}$ such that, for all $B \in \mathcal{V}$, there are infinitely many $n \in \omega$ such that the first element of B that is $\geq g(b)$ is $\leq$ the first element of A that is $\geq n$.*

Theorem 3 *Two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ satisfy $\omega \underset{analytic}{\overset{Z}{\longrightarrow}} (\mathcal{U}, \mathcal{V})^\omega$ if and only if all the following conditions are satisfied. ("And symmetric" means to interchange $\mathcal{U}$ with $\mathcal{V}$ and to interchange Z with $\omega - Z$.)*

1. *If Z contains two consecutive integers n and $n + 1$, then $\mathcal{U}$ is selective. And symmetric.*

2. *If Z is neither $\varnothing$ nor $\{0\}$, then $\mathcal{U}$ is a P-point. And symmetric.*

3. *If neither Z nor $\omega - Z$ is an initial segment of ω, then $\mathcal{U}$ and $\mathcal{V}$ are not nearly coherent.*

4. *If Z is a nonempty, proper, initial segment of ω, then either $\mathcal{U}$ and $\mathcal{V}$ are not nearly coherent or there is a finite-to-one $f : \omega \to \omega$ and there are sets $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $f(\mathcal{U}) = \mathcal{V}$ and that, whenever $a \in A$, $b \in B$, and $f(a) = b$, then $b < a$. And symmetric.*

Notice that, in item 4, $\mathcal{V}$ must be selective (by item 1) and so, if $\mathcal{U}$ and $\mathcal{V}$ are nearly coherent, then $f(\mathcal{U}) = \mathcal{V}$ for some finite-to-one f. Thus, the last alternative in item 4 amounts to one of the conditions from Theorem 2 (with $\mathcal{V}$ here playing the roles of both $\mathcal{V}$ and $\mathcal{W}$ in Theorem 2 and with the identity function in place of f as the map from $\mathcal{V}$ to $\mathcal{V}$).

Union Ultrafilters

Let $\mathbb{F}$ be the set of finite, nonempty subsets of ω. For any sequence $\mathbf{a} = \langle a_0, a_1, \ldots, a_n, \ldots \rangle$ of disjoint elements of $\mathbb{F}$, let $FU(\mathbf{a})$ be the set of nonempty finite unions of a_n's. A *union* ultrafilter is an ultrafilter on $\mathbb{F}$ with a basis of sets of the form $FU(\mathbf{a})$. That these exist under CH (or MA) is a consequence of Hindman's partition theorem for finite unions.

Write min and max for the maps $\mathbb{F} \to \omega$ sending each $a \in \mathbb{F}$ to its first and last element.

Hindman and I proved long ago (1987) that, if $\mathcal{U}$ is a union ultrafilter, then $\min(\mathcal{U})$ and $\max(\mathcal{U})$ are P-points.

An apparently new result is that they are not nearly coherent.