Labeled Oriented Intervals That Are Not Diagrammatically Reducible

Ashley Earls, Jens Harlander, Gabriel Islamboul, Rachael Keller, Mingjia Yang

1St. Olaf College, 2Boise State University, 3University of Virginia, 4Louisiana State University, 5Albion College

Introduction

Consider a set of planar tiles with labeled and oriented boundary edges. If one glues the tiles together so that the labels and the orientations match, one obtains a 2-complex. A 2-complex is called aspherical if every map from a 2-sphere into the 2-complex extends to the 3-ball. Asphericity means that the 2-complex does not have holes that are not already detected by the fundamental group.

J.H.C. Whitehead’s Conjecture (1941) states that a sub-complex of an aspherical 2-complex is aspherical. The conjecture originally arose in the context of knot theory. It implies the asphericity of knot complements. Papakyriakopoulos established asphericity of knot complements in 1956. Whitehead’s conjecture remains open to this day.

Spherical maps into 2-complexes can be encoded by spherical diagrams. A spherical diagram over a 2-complex is a 2-sphere tessellated with tiles of that 2-complex and their mirror images. A spherical diagram is called reduced if no tile is adjacent to its mirror image. A 2-complex is called diagrammatically reducible (DR) if there does not exist a reduced spherical diagram using the tiles of the 2-complex. It can be shown that a 2-complex that is DR is also aspherical.

We study Whitehead’s conjecture in the context of labeled oriented intervals (or, equivalently, long virtual knots).

Labeled Oriented Intervals

A labeled oriented graph is an oriented graph where the edges are labeled by vertices. A labeled oriented interval (LOI) is a labeled oriented graph where the underlying graph is a subdivided line. Each edge in the interval gives rise to a square with a labeled and oriented boundary. Note that each square also corresponds to a crossing (see figure below).

If we glue the squares together, respecting labels and orientations, we obtain a 2-complex. This 2-complex is called the Wirtinger complex associated with the LOI. If \( P \) denotes the LOI, then \( W(P) \) is the associated Wirtinger complex. It is not difficult to see that \( W(P) \) is a sub-complex of a contractible and hence aspherical 2-complex. Thus Wirtinger complexes are a good testing ground for Whitehead’s conjecture.

There is strong empirical evidence that Whitehead’s conjecture is true for Wirtinger complexes coming from labeled oriented trees and intervals. Three years ago, Stephan Rosebrock, a researcher at the Pädagogische Hochschule in Karlsruhe, Germany, began to investigate diagrammatic reducibility for LOIs using computers. He checked roughly 60 billion LOIs and found only a few spherical diagrams. Studying Rosebrock’s diagrams, we searched for construction patterns that make reduced spherical diagrams over certain LOIs possible. Once a pattern is recognized we can tell the computer to avoid these patterns. The ultimate goal is to find all patterns that are responsible for the existence of reduced spherical diagrams. This set of patterns could then be used for a targeted search for possible counterexamples to Whitehead’s conjecture.

Observations

The Rosebrock LOIs we investigated share many common features. Let \( P \) be a LOI from the Rosebrock list. Then the following holds:

- If \( P \) is prime, then \( G(P) \), the group defined by \( P \), is infinite cyclic;
- If \( P \) contains a sub-LOI \( Q \), then \( G(P) = G(Q) \);
- If \( P \) contains a sub-LOI \( Q \), and \( Q \) is a knot which also occurs in the reduced diagram for \( P \) cutting that knot from the diagram results in a reduced spherical picture for \( P/Q \) thus \( P/Q \) is not DR. \( Q \)'s tiles occur in the spherical diagram if and only if the tiles form \( G(Q) \)'s knot.

The figures show LOIs and reduced spherical diagrams over them. The spherical diagrams are spheres tiled by the squares of the LOIs. Shown are the dual tilings. In the process of dualizing, each square turns into a crossing and the entire spherical diagram turns into a labeled oriented link. Note that the spherical diagrams are reduced; no square is adjacent to its mirror image. The top figure shows the smallest reduced non-DR prime LOI on five vertices.

Open Questions

- Is an injective LOI always DR? Huck and Rosebrock (5) answered this question affirmatively for prime injective LOIs.
- If a prime LOI is not DR is its group always infinite cyclic? It is not true that any prime LOI with the infinite cyclic group is non-DR. We have found a LOI on three vertices with the infinite cyclic group that is DR.
- Can a LOI with consistently oriented edges be non-DR?

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References and Acknowledgements