Games and Algebraic Structures

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Motivation: Ciliates


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Signed Permutations

The number of distinct elements in $S_n$ is $n!$
On the other hand, the number of elements in $S_n^\pm$ is $2^n n!$

$$\alpha = [5 \quad -3 \quad 2 \quad -4 \quad 1]$$

$$\alpha = [(4,5)5 \quad -(4,3) - 3 -(3,2) \quad (1,2) 2(2,3) \quad -(5,4) - 4 -(4,3) \quad 1(1,2)]$$
Signed Permutations

The number of distinct elements in $S_n$ is $n!$
On the other hand, the number of elements in $S_n^\pm$ is $2^n n!$

$$\alpha = \begin{bmatrix} 5 & -3 & 2 & -4 & 1 \\ (4,5) & -(4,3) & -3 & -(3,2) & (1,2) \\ 2 & (2,3) & -(5,4) & 4 & -(4,3) & 1(1,2) \end{bmatrix}$$
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$$\alpha = \begin{bmatrix} 5 & -3 & 2 & -4 & 1 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} (4,5)5 & -(4,3) & -3 & -(3,2) & (1,2)2 & -(5,4) & -4 & -(4,3) & 1_{(1,2)} \end{bmatrix}$$
Signed Permutations

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$$\alpha = \begin{bmatrix} 5 & -3 & 2 & -4 & 1 \\ \end{bmatrix}$$

$$\alpha = \begin{bmatrix} (4,5) & -(4,3) & -(3,2) & (1,2) & 2 & (2,3) & -(5,4) & -4 & -(4,3) & 1(1,2) \end{bmatrix}$$
Context Directed Swaps

Moves defined by two pointer pairs \((n, n + 1)\) and \((m, m + 1)\)

\[
\alpha = \begin{bmatrix}
(4, 5) & (4, 3) & (3, 2) \\
5 & -3 & (1, 2) \\
(2, 3) & (5, 4) & -4 \\
& -3 & (3, 2) \\
\end{bmatrix}
\]

Consider the move defined by the pointers \((1, 2)\) and \(-(4, 3)\)

\[
\alpha_0 = \begin{bmatrix}
(4, 5) & 1 & (1, 2) & (2, 3) \\
5 & (1, 2) & (2, 3) & (5, 4) \\
& 4 & -4 & -(4, 3) \\
& & -(4, 3) & -3 \\
\end{bmatrix}
\]
Context Directed Swaps

Moves defined by two pointer pairs \((n, n + 1)\) and \((m, m + 1)\)

\[ \alpha = \begin{bmatrix} (4, 5) & - (4, 3) & - (3, 2) & (1, 2) & 2_{(2, 3)} & - (5, 4) & - 4 - (4, 3) & 1_{(1, 2)} \end{bmatrix} \]

Consider the move defined by the pointers \((1, 2)\) and \(- (4, 3)\)

\[ \alpha_0 = \begin{bmatrix} (4, 5) & 1_{(1, 2)} & (1, 2) & 2_{(2, 3)} & - (5, 4) & - 4 - (4, 3) & -(4, 3) & - 3 - (3, 2) \end{bmatrix} \]
Context Directed Swaps

Moves defined by two pointer pairs \((n, n+1)\) and \((m, m+1)\)

\[
\alpha = \begin{bmatrix}
(4, 5) & - (4, 3) - 3 - (3, 2) & (1, 2) & 2 & -(5, 4) - 4 - (4, 3) & 1 (1, 2)
\end{bmatrix}
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Consider the move defined by the pointers \((1, 2)\) and \(-(4, 3)\)

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\alpha_0 = \begin{bmatrix}
(4, 5) & 1 (1, 2) & (1, 2) & 2 & -(5, 4) - 4 - (4, 3) & -(4, 3) - 3 - (3, 2)
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\alpha_0 = \begin{bmatrix}
(4, 5) & 5 & 1 & -(4,3) & -3 & -(3,2) & (1,2) & 2 & (2,3) & -(5,4) & -4 & -(4,3) & -(4,3) & 3 & -(3,2)
\end{bmatrix}
\]
Context Directed Reversals

Moves defined by one pair \((p, p + 1)\) (and its negative \(- (p + 1, p)\).)

\[
\alpha = \begin{bmatrix}
(4,5) & - (4,3) & - (3,2) & (1,2) & 2 & (2,3) & - (5,4) & - 4 & - (4,3) & 1 \\
5 & -3 & 3 & -2 & 2 & -3 & 4 & -4 & 3 & 1
\end{bmatrix}
\]

Consider the move defined by the pointer \(- (3, 2)\).

\[
\alpha_1 = \begin{bmatrix}
(4,5) & - (4,3) & - (3,2) & - (3,2) & -2 & (2,1) & - (5,4) & - 4 & - (4,3) & 1 \\
5 & -3 & 3 & 2 & 2 & -3 & 4 & -4 & 3 & 1
\end{bmatrix}
\]
Context Directed Reversals

Moves defined by one pair \((p, p + 1)\) (and its negative \(-(p + 1, p)\).)

\[
\alpha = \begin{bmatrix}
(4, 5) & -(4, 3) & -3 & -(3, 2) & (1, 2) & 2 & -(2, 3) & -(5, 4) & -4 & -(4, 3) & 1
\end{bmatrix}
\]

Consider the move defined by the pointer \(-(3, 2)\).

\[
\alpha_1 = \begin{bmatrix}
(4, 5) & -(4, 3) & -3 & -(3, 2) & -(3, 2) & -2 & -(2, 1) & -(5, 4) & -4 & -(4, 3) & 1
\end{bmatrix}
\]
Context Directed Reversals

Moves defined by one pair \((p, p + 1)\) (and its negative \(−(p + 1, p)\).)

\[
\alpha = \begin{bmatrix}
(4,5) & 5 \quad -(4,3) & -3 \quad -(3,2) \\
(1,2) & 2 \quad (2,3) & -5 & -4 \quad -(4,3) & 1
\end{bmatrix}
\]

Consider the move defined by the pointer \(−(3, 2)\).

\[
\alpha_1 = \begin{bmatrix}
(4, 5) & 5 \quad -(4,3) & -3 \quad -(3,2) \\
-(3,2) & -2 \quad -(2,1) & -5 & -4 \quad -(4,3) & 1
\end{bmatrix}
\]
Context Directed Reversals

Moves defined by one pair \( (p, p + 1) \) (and its negative \( -(p + 1, p) \)).

\[
\alpha = \begin{bmatrix}
(4,5) & -(4,3) & -3 & -(3,2) & (1,2) & 2 & (2,3) & -(5,4) & -4 & -(4,3) & 1
\end{bmatrix}
\]

Consider the move defined by the pointer \( -(3, 2) \).

\[
\alpha_1 = \begin{bmatrix}
(4,5) & -(4,3) & -3 & -(3,2) & -(3,2) & -2 & -(2,1) & -(5,4) & -4 & -(4,3) & 1
\end{bmatrix}
\]
Invertibility

What does invertibility of a permutation mean?

- $\alpha$ a permutation has an inverse function.
- Although in general all permutations in $S_n^\pm$ are invertible, its inverse cannot always be computed using CDR/CDS.
- If the inverse can be computed using CDS/CDR then $\alpha$ is CDR/CDS-invertible.
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Imagine, an ambassador of advanced alien race is beamed down to earth. It turns out that this advanced species has the same cell structure as the ciliate and the same cellular operations. In an attempt to upgrade their genome (and improve their mathematical abilities) they have killed off millions of their citizens... The sole purpose of this trip to earth is to catalog all permutations in $\mathcal{S}_{55}^\pm$ which are sortable by their genomes, if we cannot deliver this, they will destroy us all.
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Imagine, an ambassador of advanced alien race is beamed down to earth. It turns out that this advanced species has the same cell structure as the ciliate and the same cellular operations. In an attempt to upgrade their genome (and improve their mathematical abilities) they have killed off millions of their citizens... The sole purpose of this trip to earth is to catalog all permutations in $S_{55}^\pm$, which are sortable by their genomes, if we cannot deliver this, they will destroy us all.
Big Question

So, is there a good way to go about this?

- This is only $2^{55} \cdot 55!$ cases to check... so we could begin by running through sequences of CDS and CDR for each of the cases... $55! \approx 1.8 \cdot 10^{63}$ cases per person

- Suppose we could check that many cases by hand in a reasonable amount of time, well each case has sub-cases which in turn have more sub-cases...

- One of the goals of this summer was to answer this question:
  - Given a permutation $\alpha \in S_n^{\pm}$, can we determine an initial criterion to check whether $\alpha$ is invertible without performing any moves?

- So can it be done, or must we resort to what Erdös might suggest and attempt to destroy the aliens before they destroy us?
  - **Spoiler Alert**, we can! But we must first develop a few tools...
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Graphing Signed Permutations: Vertices

We take $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$, and including "poles" and "anchors" we have

$$0 [-3^+, + - 1^-, + - 2^-, -5^+, -4^+] 6$$

The signed cycle graph is made up of one set of vertices and two sets of edges.

$V_\alpha = 0, 1^-, 1^+, 2^-, 2^+, 3^-, 3^+, 4^-, 4^+, 5^-, 5^+, 6$

Figure: $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$
Graphing Signed Permutations: Vertices

We take $\alpha = [3 -1 -2 5 4]$, and including "poles" and "anchors" we have

$$0 [-3^+, + - 1^-, + - 2^-, -5^+, -4^+] 6$$

The signed cycle graph is made up of one set of vertices and two sets of edges.

$V_\alpha = 0, 1^-, 1^+, 2^-, 2^+, 3^-, 3^+, 4^-, 4^+, 5^-, 5^+, 6$

Figure: $\alpha = [3 -1 -2 5 4]$
Graphing Signed Permutations: Vertices

We take $\alpha = [3 -1 -2 5 4]$, and including "poles" and "anchors" we have

$$0 \ [−3^+, + − 1^−, + − 2^−, −5^+, −4^+] 6$$

The signed cycle graph is made up of one set of vertices and two sets of edges.

$V_\alpha = 0, 1^−, 1^+, 2^−, 2^+, 3^−, 3^+, 4^−, 4^+, 5^−, 5^+, 6$

Figure: $\alpha = [3 − 1 − 2 5 4]$
Graphing Signed Permutations: Vertices

We take $\alpha = [3 -1 -2 5 4]$, and including "poles" and "anchors" we have

$$0 [-3^+, + - 1^-, + - 2^-, -5^+, -4^+] 6$$

The signed cycle graph is made up of one set of vertices and two sets of edges.

$V_\alpha = 0, 1^-, 1^+, 2^-, 2^+, 3^-, 3^+, 4^-, 4^+, 5^-, 5^+, 6$

Figure: $\alpha = [3 - 1 - 2 5 4]$
Graphing Signed Permutations: Vertices

We take $\alpha = [3 \; -1 \; -2 \; 5 \; 4]$, and including "poles" and "anchors" we have

$$0 \; [-3^+, \; + \; -1^-, \; + \; -2^-, \; -5^+, \; -4^+] \; 6$$

The signed cycle graph is made up of one set of vertices and two sets of edges.

$$V_\alpha = 0,1^-,1^+,2^-,2^+,3^-,3^+,4^-,4^+,5^-,5^+,6$$

Figure: $\alpha = [3 \; -1 \; -2 \; 5 \; 4]$
Graphing Signed Permutations: Vertices

We take \( \alpha = [3 \ -1 \ -2 \ 5 \ 4] \), and including "poles" and "anchors" we have

\[
0 \ [-3^+, \ + - 1^-, \ + - 2^-, \ -5^+, \ -4^+] \ 6
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The signed cycle graph is made up of one set of vertices and two sets of edges.

\( V_\alpha = 0, 1^-, 1^+, 2^-, 2^+, 3^-, 3^+, 4^-, 4^+, 5^-, 5^+, 6 \)

Figure: \( \alpha = [3 \ -1 \ -2 \ 5 \ 4] \)
Graphing Signed Permutations: Vertices

We take $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$, and including "poles" and "anchors" we have

$$0 \ [3^+ , \ +\ -1^- , \ +\ -2^- , \ -5^+ , \ -4^+] \ 6$$

The signed cycle graph is made up of one set of vertices and two sets of edges.

$$V_\alpha = 0, 1^-, 1^+, 2^-, 2^+, 3^-, 3^+, 4^-, 4^+, 5^-, 5^+, 6$$

Figure: $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$
The set of blue edges connect adjacent elements. For $\alpha = [ -3^+ + -1^- + -2^- -5^+ -4^+ ]$ we begin with the anchor $n + 1 = 6$. The positive side of 4 is to the left of 6 so $6 \rightarrow 4^+$.

Figure: $\alpha = [3^- 1^- 2^- 5^+ 4]$
Graphing Signed Permutations: Black Edges

The set of blue edges connect adjacent elements.
For \( \alpha = [ -3^+ + -1^- + -2^- 5^+ 4^+ ] \) we begin with the anchor  \( n + 1 = 6 \).
The positive side of 4 is to the left of 6 so 6 \( \rightarrow 4^+ \)

Figure: \( \alpha = [3^- 1^- 2^- 5^+ 4] \)
Graphing Signed Permutations: Black Edges

The set of blue edges connect adjacent elements.
For $\alpha = [-3^+ + -1^- + -2^- -5^+ -4^+]$ we begin with the anchor $n + 1 = 6$.

The positive side of 4 is to the left of 6 so $6 \rightarrow 4^+$

Figure: $\alpha = [3^- 1^- 2^- 5^+ 4^+]$
Graphing Signed Permutations: Black Edges

The set of blue edges connect adjacent elements. For $\alpha = [ -3^+ + -1^- + -2^- -5^+ -4^+ ]$ we begin with the anchor $n + 1 = 6$. The positive side of 4 is to the left of 6 so $6 \rightarrow 4^+$

Figure: $\alpha = [3^- 1^- 2^- 5^+ 4^-]$
Graphing Signed Permutations: Black Edges

The set of blue edges connect adjacent elements. For \( \alpha = [-3^+ - 1^- + - 2^- - 5^+ - 4^+] \) we begin with the anchor \( n + 1 = 6 \).

\[ 4^+ \rightarrow 4^- \]

Figure: \( \alpha = [3^- - 1^- - 2^+ 5^- 4^-] \)
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$$4^- \rightarrow 5^+$$

Figure: $\alpha = [3^- 1^- 2^- 5^+ 4]$
The set of blue edges connect adjacent elements. For $\alpha = [-3^+ + -1^- + -2^- -5^+ -4^+]$ we begin with the anchor $n + 1 = 6$.

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\[
2^- \rightarrow 2^+
\]

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The set of blue edges connect adjacent elements. For $\alpha = [-3^+ + -1^- + -2^- -5^+ -4^+]$ we begin with the anchor $n + 1 = 6$.

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The set of blue edges connect adjacent elements. For \( \alpha = [ -3^+ + -1^- + -2^- -5^+ -4^+ ] \) we begin with the anchor \( n + 1 = 6 \).

\[ 1^- \rightarrow 1^+ \]

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![Graph of signed permutations]

$1^+ \rightarrow 3^+$

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$3^+ \rightarrow 3^-$

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The set of blue edges connect adjacent elements.
For $\alpha = [-3^+ + - 1^- + - 2^- - 5^+ - 4^+]$ we begin with the anchor $n + 1 = 6$.

$\begin{align*}
3^- \rightarrow 0 \\
5^+ \\
5^- \\
4^+ \\
4^- \\
3^+ \\
3^- \\
1^- \\
1^+ \\
2^- \\
2^+
\end{align*}$

Figure: $\alpha = [3 - 1 - 2 5 4]$
Graphing Signed Permutations: Grey Edges

The set of red edges connect numerically consecutive elements. In canonical order, the elements are: $-1^+, -2^+, -3^+, ..., -n^+$. So we consider $1^+$ and $2^-$ to be consecutive, along with every other $x^+$ and $(x + 1)^-$. $0$ and $1^-$ are also consecutive, as are $n^+$ and $(n + 1)$.

Figure: $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$
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Figure: $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$
Alternating Cycles

Now that we have a cycle graph, we can walk an alternating cycle on it!

\[ \alpha = [3 \ -1 \ -2 \ 5 \ 4] \]

Figure: Alternating cycle on a cycle graph, with alternating edges labeled as + and -.
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Figure: $\alpha = [3 - 1 - 2 5 4]$

$0 \cdots 1^{-} \leftarrow 2^{+} \cdots 3^{-} \rightarrow 0$
Alternating Cycles

And another one!

Figure: $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$
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Figure: $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$

$1^+ \cdots 2^- \leftarrow 5^-$
Alternating Cycles

And another one!

Figure:  \( \alpha = [3 \ -1 \ -2 \ 5 \ 4] \)

\[ 1^+ \cdots 2^- \leftarrow 5^- \cdots 4^+ \]
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And another one!

Figure: \( \alpha = [3 \ -1 \ -2 \ 5 \ 4] \)

\[ 1^+ \cdot 2^- \leftarrow 5^- \cdot 4^+ \leftarrow 6 \]
Alternating Cycles

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Figure: $\alpha = [3 \ -1 \ -2 \ 5 \ 4]$

$1^+ \cdots 2^- \leftarrow 5^- \cdots 4^+ \leftarrow 6 \cdots 5^+$
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And another one!

Figure: $\alpha = [3 - 1 - 2 5 4]$

$1^+ \ldots 2^- \leftarrow 5^- \ldots 4^+ \leftarrow 6 \ldots 5^+ \leftarrow 4^- \ldots 3^+$
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Figure: $\alpha = [3\ -\ 1\ -\ 2\ 5\ 4]$
Facts about CDS/CDR

- Since CDS/CDR are restricted moves, guided by pointers they necessarily “repair” breakpoints.
  - CDS “repairs” 2-4 breakpoints per move.
  - CDR “repairs” 1-2 breakpoints per move.

- When moves are performed on $\alpha$ how does this affect $\alpha$’s alternating cycles?
  - Breakpoints are “repaired”, and a “short-circuiting” occurs.
  - Basically, we have that when CDS or CDR are applied to a permutation, elements of the alternating cycles drop out and no elements are added.

Now that we have added some tools, we recall our main question:
- Can we determine an initial criterion to determine invertibility?
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Theorem (Invertibility Criterion)

Let $\alpha \in S_n^\pm$.

$\alpha$ is invertible by CDR/CDS if, and only if, the elements $(\rightarrow 0 \cdots 1^-)$ and $(n \cdots (n + 1)^+ \rightarrow)$ do not appear in the same alternating cycle in the order:

$$\cdots \rightarrow 0 \cdots 1^- \cdots n^+ \cdots n + 1 \rightarrow \cdots$$

- Now, instead of running through all possible moves on a permutation, we need only examine alternating cycles.
- Why is this important?
  - We know precisely which permutations are, and are not invertible by our operations, and in turn by the ciliates.
  - The world is now safe from all aliens demanding to know CDS/CDR invertible permutations!
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Implications to Games

What on earth does this have to do with Game Theory?

We now know which permutations can be ordered by our operations. But if the aliens return demanding more information, we want to be prepared! Certain facts about our restricted operations lend themselves well to games:

- CDR/CDS uninvertible permutations may produce different end states under different paths of moves.
- CDR invertible permutations only reach an "inverted" state under certain paths.

So we can construct games that rely on these changing outcomes in order to examine the effect of following different paths of moves.
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This leads us to...

The Fundamental Theorem of Finite Games (Zermelo)
Every two-player, win-lose, perfect information game is determined.

In a determined game, some player has a winning strategy.

Given that we can predict CDS and CDR invertibility, can we build games on CDS and CDR with winning strategies?
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Building Games

The creation of a game on our permutations comes from:

- Choosing the available moves (only CDS, only CDR, or either)
- Specifying a winning condition

CDS games: In these games only CDS swaps are allowed. The game lasts until there are no available moves.

- If the final position of $\alpha$ is ordered (the identity permutation), then Player ONE wins. Else, TWO wins.
- Player ONE wins if $k < \frac{n}{2}$ for non-CDS-invertible $\alpha$ in the final state $[k + 1 \ldots n 1 \ldots k]$. If $\alpha$ is CDS-invertible, ONE wins also. Otherwise, TWO wins.
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Are these games determined?

From the Fundamental Theorem of Finite Games, these games need to be:

- Finite,
- Perfect Information,
- Win-lose,

in order to be determined.
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Recall from the explanation of the use of pointers in CDS and CDR that each application requires one or two pointer pairs as in:

$$\alpha = \left[ \begin{array}{c}
(4,5) & 5 & -(4,3) & -3 & -(3,2) & (1,2) & 2 & (2,3) & -(5,4) & -4 & -(4,3) & 1
\end{array} \right]$$

which becomes

$$\alpha_0 = \left[ \begin{array}{c}
(4,5) & 5 & 1 & (1,2) & (1,2) & 2 & (2,3) & -(5,4) & -4 & -(4,3) & -(4,3) & 3 & -(3,2)
\end{array} \right]$$

Also, $\alpha$ has a finite number of terms and pointers. Every move requires the use of these pointers, which cannot be used a second time, so there must be a finite number of moves.

As game play ends when there are no more moves, the game is finite.

Since each player observes the opponent’s moves, each player has perfect information.

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which becomes

\[ \alpha_0 = \begin{bmatrix} (4,5) & 5 & 1^+ & (1,2)^+ & (1,2)^+ & 2^+ & (2,3)^- & (5,4)^- & 4^- & (4,3)^- & (4,3)^- & 3^- & (3,2)^- \end{bmatrix} \]

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Implications

- Each of these games is determined.
- In each of these games, some player has a winning strategy.
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The Big Question

So now we have some determined games! Someone must win, leading us to ask to the most important question in any game...
Can I win? (And how?)
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Defining the Strategic Pile

Consider games played on CDS-CDR uninvertible permutations. Which ending conditions are possible? To find out, we examine alternating cycles that are defined using the cycle graph.

For example, \( \alpha = [-6 \ 5 \ 4 \ 7 \ -1 \ 2 \ 8 \ 3] \)

The cycle graph of \( \alpha \) contains one alternating cycle:

\[
\begin{align*}
1^- & \leftrightarrow 2^- & 1^+ & \rightarrow 7^+ & 8^- & \rightarrow 2^+ & 3^- \\
0 & \uparrow & & & & & & & & & & & & & & 8^+ \\
6^+ & \downarrow & & & & & & & & & & & & & & 9 \\
7^- & \rightarrow 4^+ & 5^- & \rightarrow 6^- & 5^+ & \leftrightarrow 4^- & 3^+ & \leftrightarrow 
\end{align*}
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\begin{array}{ccccccccccc}
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\end{array}
\]
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\( \alpha \) is \textbf{uninvertible} by CDR, CDS, or CDR and CDS.
Defining the Strategic Pile

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We call the set of pairs of the form \( k^+ \cdots (k + 1)^- \) between \( n + 1 \) and 0 the **Strategic Pile**.

\[
\begin{align*}
0 & 
\rightarrow & 
2^- & 
\rightarrow & 
1^+ & 
\rightarrow & 
7^+ & 
\rightarrow & 
8^- & 
\rightarrow & 
2^+ & 
\rightarrow & 
3^- & 
\rightarrow & 
8^+ & 
\rightarrow & 
9 \\
6^+ & 
\rightarrow & 
7^- & 
\rightarrow & 
4^+ & 
\rightarrow & 
5^- & 
\rightarrow & 
6^- & 
\rightarrow & 
5^+ & 
\rightarrow & 
4^- & 
\rightarrow & 
3^+ & 
\rightarrow & 
6^+
\end{align*}
\]

\( \alpha \)'s strategic pile is:

\[
\{ 3^+ \cdots 4^-, 5^+ \cdots 6^-, 5^- \cdots 4^+, 7^- \cdots 6^+ \} 
\]
We call the set of pairs of the form $k^+ \cdots (k + 1)^-$ between $n + 1$ and 0 the **Strategic Pile**.

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We call the set of pairs of the form $k^+ \cdots (k + 1)^-$ between $n + 1$ and 0 the **Strategic Pile**.

\[ \{3^+ \cdots 4^-, 5^+ \cdots 6^-, 4^- \cdots 5^+, 6^- \cdots 7^+\} \]
The Elements of the Strategic Pile

We can’t remove all elements between \( n + 1 \) and 0 since they cannot be adjacent to each other.
So once all possible moves are made, there must be one element left in the strategic pile.

Suppose that element is \( 5^- \cdots 4^+ \).
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Suppose that element is \( 5^- \ldots 4^+ \).
We can’t remove all elements between \( n + 1 \) and 0 since they cannot be adjacent to each other. So once all possible moves are made, there must be one element left in the strategic pile.

Suppose that element is \( 5^- \cdots 4^+ \).

\[
\begin{array}{c}
1^- & 2^- & 1^+ & 7^+ & 8^- & 2^+ & 3^- & 8^+\\
0 & 1^+ & 7^+ & 8^- & 2^+ & 3^- & 8^+ & 9\\
6^+ & 7^- & 4^+ & 5^- & 6^- & 5^+ & 4^- & 3^+ & 3^-
\end{array}
\]
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Suppose that element is $5^- \cdots 4^+$. 
The Elements of the Strategic Pile

From the alternating cycle:

\[
\begin{array}{c}
\begin{array}{c}
1^-
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
8^+
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
9
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
4^+
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
5^-
\end{array}
\end{array}
\end{array}
\]

We can construct:

\[
\begin{array}{c}
\begin{array}{c}
[+4 \ldots \quad \ldots]
\end{array}
\end{array}
\downarrow
\begin{array}{c}
\begin{array}{c}
[+4 \ldots \quad \ldots 5^-]
\end{array}
\end{array}
\downarrow
\begin{array}{c}
\begin{array}{c}
[-4 \ldots \quad \ldots -5]
\end{array}
\end{array}
\downarrow
\begin{array}{c}
\begin{array}{c}
[-4 \quad -3 \quad -2 \quad -1 \quad -8 \quad -7 \quad -6 \quad -5]
\end{array}
\end{array}
\end{array}
\]
The Elements of the Strategic Pile

From the alternating cycle:

\[\begin{array}{c}
0 \\
1^- \\
4^+ \\
5^- \\
8^+ \\
9 \\
\end{array}\]

We can construct:

\[
\begin{array}{c}
[+4\ldots] \\
[+4\ldots] \\
[+4\ldots] \\
[-4\ldots] \\
[-4 - 3 - 2 - 1 - 8 - 7 - 6 - 5] \\
\end{array}\]
The Elements of the Strategic Pile

From the alternating cycle:

\[
\begin{array}{c}
0 & 1^- & 8^+ & 9 \\
4^+ & 5^- &
\end{array}
\]

We can construct:

\[
\begin{array}{cccc}
[+4... & \ldots] \\
\downarrow & & & \\
[+4... & \ldots5^-] \\
\downarrow & & & \\
[\ldots5^- & \ldots] \\
\downarrow & & & \\
[-4... & \ldots5] \\
\downarrow & & & \\
[-4 & \ldots5]
\end{array}
\]
The Elements of the Strategic Pile

From the alternating cycle:

\[\begin{array}{c}
1^- & 8^+ \\
0 & 9 \\
4^+ & 5^-
\end{array}\]

We can construct:

\[\begin{array}{c}
[+4,...] \\
\downarrow \\
[+4,...] \\
\downarrow \\
[-4,...] \\
\downarrow \\
[-4 - 3 - 2 - 1 - 8 - 7 - 6 - 5]
\]
The Elements of the Strategic Pile

From the alternating cycle:

\[
\begin{align*}
0 &\rightarrow 1^- &\rightarrow 8^+ &\rightarrow 9 \\
4^+ &\rightarrow 5^- &\rightarrow 4^+ &\rightarrow 5^-
\end{align*}
\]

We can construct:

\[
\begin{align*}
[+4... &\rightarrow ...5^-] \\
[-4... &\rightarrow ...5^-] \\
[-4 - 3 - 2 - 1 - 8 - 7 - 6 - 5]
\end{align*}
\]
The Elements of the Strategic Pile

From the alternating cycle:

\[ +4 \ldots \]
\[ \downarrow \]
\[ +4 \ldots \]
\[ \downarrow \]
\[ -4 \ldots \]
\[ \downarrow \]
\[ -4 \quad -3 \quad -2 \quad -1 \quad -8 \quad -7 \quad -6 \quad -5 \]

We can construct:

\[ +4 \ldots \]
\[ \downarrow \]
\[ +4 \ldots \]
\[ \downarrow \]
\[ -4 \ldots \]
\[ \downarrow \]
\[ -5 \]
The Elements of the Strategic Pile

From the alternating cycle:

\[\begin{align*}
0 & \rightarrow 1^- & 8^+ & \rightarrow 9 \\
& \rightarrow 4^+ & 5^- & \rightarrow
\end{align*}\]

We can construct:

\[\begin{align*}
[+4... & \ldots] \\
\downarrow \\
[+4... & \ldots 5^-] \\
\downarrow \\
[-4... & \ldots - 5] \\
\downarrow \\
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8^+ \\
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\[
\begin{bmatrix}
[+4... ] \\
\downarrow \\
[+4... ] \\
\downarrow \\
[-4... ] \\
\downarrow \\
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\end{bmatrix}
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0 & 9 \\
&4^+ & 5^- \\
\end{align*}\]

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\[
\begin{bmatrix}
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\downarrow & \downarrow \\
[+4\ldots & \ldots5^-] \\
\downarrow & \downarrow \\
[-4\ldots & \ldots - 5] \\
\downarrow \\
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\end{bmatrix}
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The Elements of the Strategic Pile

From the alternating cycle:

$$\begin{array}{ccccccccc}
+4 & \cdots & & & & & \\
\downarrow & & & & & & \\
+4 & \cdots & & & & & \\
\downarrow & & & & & & \\
-4 & \cdots & & & & & \\
\downarrow & & & & & & \\
-4 & -3 & -2 & -1 & -8 & -7 & -6 & -5 & \\
\end{array}$$
The Elements of the Strategic Pile

So when $5^- \cdots 4^+$ is the only Strategic Pile element left, the fixed condition that results is $[-4 - 3 - 2 - 1 - 8 - 7 - 6 - 5]$. The other elements of the pile, $3^+ \cdots 4^-, 5^+ \cdots 6^-, 7^- \cdots 6^+$, result in the ending conditions:

$$
\begin{align*}
[ & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 ] \\
[ & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 ] \\
[-6 & -5 & -4 & -3 & -2 & -1 & -8 & -7 ]
\end{align*}
$$
The Elements of the Strategic Pile

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\end{bmatrix}
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The Elements of the Strategic Pile

In general, we consider the element $k^- \cdots (k + 1)^+$ to correspond to the fixed condition:

- $[(k + 1) \ldots n \ 1 \ldots k]$ if the alternating cycle is of the form $n + 1 \rightarrow \ldots k^+ \cdots (k + 1)^- \ldots$

- $[-k \ldots -1 \ -n \ldots -(k + 1)]$ if the alternating cycle is of the form $\ldots n + 1 \rightarrow \ldots (k + 1)^- \cdots k^+ \ldots$
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- $[−k\ldots−1\ − n\ldots−(k + 1)]$ if the alternating cycle is of the form $\ldots n + 1 \rightarrow \ldots (k + 1)^{−}\cdots k^+$ $\ldots$
The Elements of the Strategic Pile

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The Elements of the Strategic Pile

1. The strategic pile can be partitioned into a subpile of elements corresponding to "wins" for ONE and a subpile corresponding to "wins" for TWO, according to the specified winning conditions.

2. When a move is made including some pointer \((i, i + 1), i^+ \cdots (i + 1)^-\) is removed from any larger alternating cycle.

3. So a general tactic to increase your chances of winning is to make moves that remove elements from your opponent's subpile.
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Winning an "Uninvertible" Game

Consider the game where ONE and TWO alternately choose CDS moves on a CDS-uninvertible permutation of positive numbers until there is no legal move left. ONE wins if \( k < \frac{n}{2} \) in the final state \( \alpha = [k+1...n 1...k] \). Otherwise, TWO wins.

- If all the elements are removed from TWO’s subpile before all of the elements are removed from ONE’s subpile, ONE wins.
- There always exists a move that removes:
  - Two elements from your opponent’s subpile OR
  - One element from your subpile and one element from your opponent’s subpile
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Winning an "Uninvertible" Game

Consider the worst case scenario for ONE in such a game:

- On ONE’s turn they must remove an element from their own subpile in order to remove one from TWO’s subpile.
- On TWO’s turn, TWO can remove two elements from ONE’s subpile.
- So in the game as a whole, three times more elements are removed from ONE’s subpile than from TWO’s subpile.

So if 3/4 of the elements in the Strategic Pile are in ONE’s subpile, ONE has a winning strategy. A winning strategy for TWO can be found similarly, accounting for the fact that ONE always makes the first move. If TWO has two more than 3/4 of the elements in their subpile then TWO has a winning strategy.
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A game played on a permutation that is invertible by CDS, CDR, or CDS and CDR does not have a Strategic Pile, so finding a winning strategy is more difficult. But we do have a few insights:

- If $\alpha$ is CDS invertible, the player attempting to put the elements in canonical order through CDS will always win.
- If $\alpha$ is CDS-CDR invertible, the player attempting to put the elements in canonical (or reverse canonical) order through CDS and CDR will always win.
- If $\alpha$ is CDR forward invertible, the player attempting to reach a positive ending condition through CDR (or CDR and CDS) will always win.
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Open Questions/Future Work

- Characterize which player has a winning strategy in our games.
  - CDS games where neither player has 3/4 of the Strategic Pile in their subpile
  - Path dependent invertible CDR games
- Determine if the group theoretic order of permutations are related to invertibility by CDS or CDR.
- Prove that the possible end states in CDR games correspond bijectively to elements in the strategic pile.
- Characterize what types of scrambled conditions occur, in which permutations, and by which paths.
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  - Path dependent invertible CDR games

- Determine if the group theoretic order of permutations are related to invertibility by CDS or CDR.

- Prove that the possible end states in CDR games correspond bijectively to elements in the strategic pile.

- Characterize what types of scrambled conditions occur, in which permutations, and by which paths.
Open Questions/Future Work

- Characterize which player has a winning strategy in our games.
  - CDS games where neither player has 3/4 of the Strategic Pile in their subpile
  - Path dependent invertible CDR games
- Determine if the group theoretic order of permutations are related to invertibility by CDS or CDR.
- Prove that the possible end states in CDR games correspond bijectively to elements in the strategic pile.
- Characterize what types of scrambled conditions occur, in which permutations, and by which paths.
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Thank you!