The Sortability of Graphs and Matrices under Context Directed Swaps

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Introduction: Ciliates

Ciliates, unicellular organisms, sort the genome of their micronucleus to assemble a transcriptionally functional macronucleus. The study of sorting algorithms helps in understanding the mechanisms by which this process occurs.

Future Work

- Further analyze and classify unsortable permutations, graphs, and matrices.
- Extend the overlap graph to include additional relationships.

CDS on the Overlap Graph (GCDS)

The overlap graph of a permutation \( \pi \), \( OG(\pi) \), has as vertices the edges of \( BG(\pi) \), each labeled by the pointer supporting the corresponding edge in \( BG(\pi) \). Two vertices of \( OG(\pi) \) are adjacent if the respective edges of \( BG(\pi) \) intersect.

For vertices \( x \) and \( y \), let \( f_x(y) \) be 1 if \( x \) and \( y \) are adjacent and 0 otherwise. For any adjacent vertices \( p \) and \( q \), let \( gcds(p,q)(OG(\pi)) \) be the graph \( OG(\pi) \) with the same vertices as \( OG(\pi) \), where for any vertices \( u \) and \( v \), \( OG^f(\pi) \) includes the edge \( (u,v) \) if and only if

\[
 f_p(u)f_q(v) + f_q(u)f_v(p) + f_u(v) \equiv 1 \pmod{2}.
\]

A graph is gcds-sortable if some sequence of gcds operations removes all of its edges.

Generalized Parity Cuts

Extending the definition of parity cut given in [2], let a generalized parity cut of a graph be a set \( S \) of vertices such that, for each vertex in the graph, the number of edges between that vertex and vertices in \( S \) is even.

Theorem 1. A two-rooted graph with roots \( x \), \( y \) is gcds-sortable if and only if only for each of \( x \), \( y \) there is a generalized parity cut containing that root and not the other.

The set of generalized parity cuts of a graph corresponds to the kernel of the adjacency matrix of the graph.

Theorem 2. A matrix is gcds-sortable if and only if its kernel contains an element \( \vec{x} \) such that \( x_0 = 0 \) and \( x_n = 1 \) and an element \( \vec{y} \) such that \( y_1 = 1 \) and \( y_0 = 0 \).

Theorem 3. The number of gcds-sortable two-rooted graphs on \( n \) vertices is

\[
 \frac{|S|}{2^{\binom{n}{2}}} \prod_{i=0}^{n-1} \left( \frac{2^{n+i} - 1}{2^{n+i} - 1} \right).
\]

Let \( r_n \) be the proportion of graphs on \( n \) vertices that are gcds-sortable. Then, the sequences \( \{r_{2n}\} \) and \( \{r_{2n+1}\} \) converge, and

\[
 \lim_{n \to \infty} r_{2n} \approx 0.2272 \quad \text{and} \quad \lim_{n \to \infty} r_{2n+1} \approx 0.1061.
\]

Let \( G \) be a graph on the vertices \( (1,2), (2,3), \ldots, (n-1,n) \) with adjacency matrix \( A \) taken over the field \( \mathbb{F}_2 \). Then, the number of ways we can add two roots to \( G \) to form a gcds-sortable graph is \( \text{gcds}(A) \).

CDS and Breakpoint Graph

In the positive permutation \( \pi \), assign to each entry \( a \) a left pointer \( (a-1,a) \) and a right pointer \( (a,a+1) \). Let

\[
 \pi = \{\alpha_1 p \} \{\alpha_2 q \} \{\alpha_3 p \} ... \{\alpha_n q \},
\]

where each \( \alpha_i \) is a block of the permutation and \( p = (x,x+1) \) and \( q = (y,y+1) \) are some pointers in the permutation. As in [1], let the context directed swap on \( \pi \) with context \( p \) and \( q \), \( cdsp(q)(\pi) \), be

\[
 \{\alpha_1 p \} \{\alpha_2 q \} \{\alpha_3 p \} ... \{\alpha_n q \}.
\]

The breakpoint graph of \( \pi \), \( BG(\pi) \), has the pointers of \( \pi \) as vertices and an edge between each pair of identical pointers.

Results

References

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