On this test you are allowed your writing instrument and the test paper. The test paper has a style sheet for propositional logic and axioms for formal arithmetic attached. You may tear these off for easy reference. Cell phones must be turned off and out of sight.

There are seven proofs on this exam. Of these, you need to do two propositional proofs (problems 1-3), two formal arithmetic proofs (problems 4-6) and problem 7, which is perhaps harder and only counts 3 points as opposed to 10 points for each of the other four required proofs. If you do all three proofs in a category, your best work will count. I do not suggest doing all three proofs on one of the two categories before you have done two in the other.

In all proofs, the justification of a line (assumption, lemma) should include references to other lines (assumptions, lemmas) it depends on and where appropriate the names of rules, axioms or theorems used. You are expected to state goals (things you are trying to prove) where appropriate and clearly distinguish them from assumptions or lemmas (things you know and can use).

You will be handed two copies of the test. One of these you do in class; the other you may do at home and hand in on Monday. The take-home version of the test is optional; if you do it it will count as a homework assignment and may at my discretion be used to adjust the test grade itself. You are expected to do it entirely by yourself; evidence that people have done it together will affect my willingness to use it to award any adjustments on the test (for anyone, not just those individuals).
1 Propositional Logic

1. General remark for all propositional logic problems: the conditions for doing these problems are identical to those on Homework 1. You may not use substitutions of equivalent expressions: use the rules and strategies on the style sheet only.

Write a formal proof of

\[((P \rightarrow Q) \land (Q \rightarrow R)) \rightarrow (P \rightarrow R)\]

in the style taught in class.

Assume (1): \((P \rightarrow Q) \land (Q \rightarrow R)\)

Goal: \(P \rightarrow R\)

Assume (2): \(P\)

Goal: \(R\)

Lemma (3): \(P \rightarrow Q\)

Lemma (4): \(Q \rightarrow R\)

Lemma (5): \(Q\), by (2), (3) modus ponens. (2) (1) m.p. does not work; you need to break a conjunction apart to use its parts. I did accept references like (1) \(P \rightarrow Q\) here, which made it clear that you were using part of the conjunction not the whole thing.

Conclusion: \(R\), by (5), (4) m.p. Once again, (5) (1) mp is not really satisfactory. This completes the proof, as this is our goal.
2. Write a formal proof of

\[(\neg A \lor \neg B) \rightarrow \neg (A \land B)\]

in the style taught in class.

**Assume (1):** \(\neg A \lor \neg B\) An assumption which is a disjunction should suggest that a proof by cases is going to happen.

**Goal:** \(\neg (A \land B)\)

**Assume (2):** \(A \land B\)

**Goal:** \(\bot\)

**Comment:** We now prove by cases using assumption (1).

**Case 1:** Assume (1a): \(\neg A\). Goal remains \(\bot\).

**Lemma (3a):** \(A\), from (2).

**Conclusion of Case 1:** \(\bot\), which is our goal, from (1a) and (3a).

**Case 2:** Assume (1b): \(\neg B\) Goal remains \(\bot\).

**Lemma (3b):** \(B\), from (2).

**Conclusion of Case 2:** \(\bot\), which is our goal, from (1b) and (3b). Since both cases have been handled, the proof is complete.

Alternative proof plan not involving cases, which some of you did:

**Assume (1):** \(\neg A \lor \neg B\)

**Goal:** \(\neg (A \land B)\)

**Assume (2):** \(A \land B\)

**Goal:** \(\bot\)

**Lemma (3):** \(A\) from (2)

**Lemma (4):** \(B\) from (2)

**Lemma (5):** \(\neg B\) from (1) and (3) by disjunctive syllogism.

Strictly speaking, the form of disjunctive syllogism that I give does not justify this: an application of double negation is needed (replace \(A\) with \(\neg \neg A\) and it is perfect). But I accepted this.

**Conclusion:** \(\bot\), which is our goal, from (4) and (5).
3. Write a formal proof of

$$\neg(A \lor B) \rightarrow (\neg A \land \neg B)$$

in the style taught in class.

**Assume (1):** $\neg(A \lor B)$

**Goal:** $\neg A \land \neg B$

**Comment:** The proof of a conjunction breaks apart into two proofs, one for each of the parts of the statement. There are no new assumptions and these are not cases. (There are assumptions introduced inside the proofs of the two parts, but proving a goal which is an and statement does not cause one to introduce any assumptions).

**Goal 1:** $\neg A$

**Assume (2):** $A$

**Goal:** $\bot$

**Comment:** A contradiction is the assertion of a statement and its exact negation. (1) and (2) are not formally in contradiction. They can't both be true, but that is what we are trying to prove! What we need for a contradiction to (1) is $A \lor B$, which we can easily get. The second part can be handled in the same way.

**Lemma (3):** $A \lor B$, by (2) and rule of addition.

**Conclusion:** $\bot$, by (1) and (3). We are done with the proof of Goal 1.

**Goal 2:** $\neg B$

**Assume (2):** $B$

**Goal:** $\bot$

**Comment:** This goes in the same way as the other part.

**Lemma (3):** $A \lor B$, by (2) and rule of addition.

**Conclusion:** $\bot$, by (1) and (3). We are done with the proof of Goal 2, and so with the entire proof.
2 Propositional Logic Style Sheet

Not all of these are relevant to the assigned proofs; you need to recognize which rules and strategies are appropriate.

**Conjunction (and):** To prove $A \land B$, prove $A$ (part 1), then prove $B$ (part 2). The parts are not cases, and you will lose credit if you call something a proof by cases which isn’t one.

From $A \land B$, deduce $A$. From $A \land B$, deduce $B$. You need to explicitly break apart assumptions or lemmas which are “and” statements to use their components; you will lose credit if you don’t.

**Disjunction (or):** To prove $A \lor B$, assume $\neg A$ and adopt the new goal $B$ [or assume $\neg B$ and adopt the new goal $A$; you do not need to do both].

From $A$, deduce $A \lor B$. From $B$, deduce $A \lor B$ (rule of addition).

From $A \lor B$ and $\neg A$, deduce $B$. From $A \lor B$ and $\neg B$, deduce $A$. This is disjunctive syllogism.

To deduce a conclusion $C$ from an assumption or lemma $A \lor B$, use *proof by cases*: in the first part (case 1) assume $A$ and prove $C$; in the second part (case 2) assume $B$ and prove $C$.

**Implication (if):** To prove $A \rightarrow B$, assume $A$ and adopt the new goal $B$.

alternative strategy: to prove $A \rightarrow B$, assume $\neg B$ and adopt the new goal $\neg A$.

Given $A$ and $A \rightarrow B$, deduce $B$ (*modus ponens*).

Given $\neg B$ and $A \rightarrow B$, deduce $\neg A$ (*modus tollens*).

If you have an assumption $A \rightarrow B$ you may want to try proving $A$ (so that you can further conclude $B$).

**Negation (not):** To prove $\neg A$, assume $A$ and try to prove $\bot$ (*contradiction*).

From $\neg \neg A$ deduce $A$ (double negation).

To prove any statement $A$, assume $\neg A$ and try to prove $\bot$. This is proof by contradiction.

From $A$ and $\neg A$, deduce $\bot$.  

5
If you have a negative assumption $\neg A$, the commonest way to use it is to wait until your goal is a contradiction, then try to prove $A$ to get the contradiction.

3 Axioms and Theorems of Formal Arithmetic

Here are the axioms of formal arithmetic.

1. 0 is a natural number (in symbols, $0 \in \mathbb{N}$).

2. If $x$ and $y$ are natural numbers, so are $S(x)$, $x + y$, and $x \cdot y$. $(\forall xy \in \mathbb{N}. S(x) \in \mathbb{N} \land x + y \in \mathbb{N} \land x \cdot y \in \mathbb{N})$.

3. 0 is not a successor. $(\forall x. S(x) \neq 0)$. Here we understand that $x \neq y$ abbreviates $\neg x = y$. Here and in the following axioms we write our quantifiers unrestricted: we could write $(\forall x \in \mathbb{N}. S(x) \neq 0)$ instead, but in this context we are only talking about natural numbers, so we can leave the restriction on our quantifiers implicit.

4. Numbers with the same successor are the same. $(\forall xy. S(x) = S(y) \rightarrow x = y)$.

5. Let $P(x)$ be any sentence about a natural number variable $x$. We assert $P(0) \land (\forall y. P(y) \rightarrow P(S(y))) \rightarrow (\forall x. P(x))$. This is a symbolic presentation of the familiar principle of mathematical induction. From an extremely technical standpoint, this is an infinite collection of axioms, one for each sentence $P(x)$. If we are also willing to talk about sets of natural numbers, we can state it as a single axiom: $(\forall A \in \mathcal{P}(\mathbb{N}). 0 \in A \land (\forall y \in \mathbb{N}. y \in A \rightarrow S(y) \in A) \rightarrow A = \mathbb{N})$. We will not use the set formulation now but we might use it later. $\mathcal{P}(\mathbb{N})$ is a notation for the collection of all sets of natural numbers.

6. $(\forall x. x + 0 = x)$

7. $(\forall xy. x + S(y) = S(x + y))$

8. $(\forall x. x \cdot 0 = 0)$

9. $(\forall x. x \cdot S(y) = x \cdot y + x)$ Here we assume the usual order of operations.
Here are some theorems and a definition.

**definition of 1**: 1 is defined as \( S(0) \).

**Theorem 1**: \((\forall x.x + 1 = S(x))\)

**Theorem 2**: \((\forall x.x = 0 \lor (\exists y.S(y) = x))\)

**Lemma 1** (for commutativity of addition): \((\forall x.0 + x = x)\)

**Lemma 2** (for commutativity of addition): \((\forall xy.S(x) + y = S(x + y))\)

**commutativity of addition**: \((\forall xy.x + y = y + x)\)

**associativity of addition**: \((\forall xyz.(x + y) + z = x + (y + z))\)