Sequent Rules for Sets (with an assignment)

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Sequent rules associated with the membership relation are the topic of this note.

Simplest are the restricted quantifier rules.

To prove the conclusion $\Gamma \vdash (\forall x \in A.\phi[x]), \Delta$, prove $\Gamma, a \in A \vdash \phi[a], \Delta$, where a is a new variable (an arbitrary object just introduced).

To prove the conclusion $\Gamma, (\forall x \in A.\phi[x]) \vdash \Delta$, prove $\Gamma, (\forall x \in A.\phi[x]) \vdash t \in A, \Delta$ and $\Gamma, \phi[t], (\forall x \in A.\phi[x]) \vdash \Delta$ for some term t.

To prove the conclusion $\Gamma \vdash (\exists x \in A.\phi[x]), \Delta$, prove $\Gamma \vdash t \in A, (\exists x \in A.\phi[x]), \Delta$ and $\Gamma \vdash \phi[t], (\exists x \in A.\phi[x]), \Delta$ for some term t.

To prove the conclusion $\Gamma, (\exists x \in A.\phi[x]) \vdash \Delta$, prove $\Gamma, a \in A, \phi[a] \vdash \Delta$ where a is a new variable.

These can be checked by writing out the definitions of restricted quantifiers in Marcel or your system and just expanding them: these are actually derived rules.

Now the comprehension rules. For these rules you need the syntactical restriction that x may not appear in the bounding set term A.

To prove $\Gamma \vdash t \in \{x \in A \mid \phi[x]\}, \Delta$, prove $\Gamma \vdash t \in A, \Delta$ and prove $\Gamma \vdash \phi[t], \Delta$. Here t can be any term. The bound variable x cannot appear in A.

To prove $\Gamma, t \in \{x \in A \mid \phi[x]\} \vdash \Delta$, prove $\Gamma, t \in A, \phi[t] \vdash \Delta$.

To prove $\Gamma \vdash t \in \mathcal{P}(A), \Delta$, prove $\Gamma, a \in t \vdash a \in A, \Delta$, where a is a new variable.

To prove $\Gamma, t \in \mathcal{P}(A) \vdash \Delta$, prove $\Gamma, u \in A, t \in \mathcal{P}(A) \vdash \Delta$ and $\Gamma, t \in \mathcal{P}(A) \vdash u \in t, \Delta$ for some term u.

The extensionality rules (these are phrased here as left and right rules for equality and I think they are complete though not easy to use: you will want rewrite style rules).

To prove $\Gamma \vdash t = u, \Delta$, prove $\Gamma, a \in t \vdash a \in u, \Delta$ and $\Gamma, a \in u \vdash a \in t, \Delta$, where a is a new variable.

To prove $\Gamma, t = u \vdash \Delta$, prove $\Gamma, u \in v, t = u \vdash \Delta$ and $\Gamma, t = u \vdash t \in v, \Delta$, where v is any term.

The previous rule may seem rather mysterious: the idea is that we can *define* $t = u$ as $(\forall v.t \in v \rightarrow u \in v)$: the rule just expands this definition.

Your assignment: verify that these rules are valid to the best of your ability (or identify my errors and correct them; I'll take another look later), and develop the rules for the constructions $\{x, y\}$ and $\bigcup A$, which are also needed for Zermelo set theory. By "valid" I mean valid in Zermelo set theory. If you need direction on this I can supply some.

If you want to be really adventurous, work out the rules for the set N of natural numbers. 0 is defined as $\emptyset$; $n+1$ is defined as $n \cup \{n\}$ (a weaker definition would be $n+1 = \{m \in \mathcal{P}(n) \mid m \in n \vee m = n\}$; this definition doesnt require existence of binary unions, which need pairs). N is defined as the intersection of all sets which contain 0 and are closed under successor. The rules for N will look like mathematical induction. This last is not assigned but you would eventually want it.