Mathematics in Three Types, 

or, doing without ordered pairs

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These are slides for a talk given to the Cameleon meeting at Cambridge University on November 24, 2005. Some corrections and additions have been made after the fact.

In particular, I have systematically made a typographical distinction between relations and functions defined by formulas and relations and functions actually coded by sets which I suggested during the talk but which was not effectively made in the original slides.

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Aims of this talk

We plan to discuss mathematical constructions, notably the definition of relations, functions, and cardinals, in the simple theory of types using only three types. This is somewhat tricky because the usual definitions of the ordered pair require at least 3 types to define a pair, and so 4 types before any implementation of a relation or a function as a set of ordered pairs is feasible. Nonetheless, quite a lot can be done.
One reason to be interested in this is that one of the fragments of Quine’s “New Foundations” which is known to be consistent is $NF_3$, shown to be consistent by Grishin, in which only those instances of comprehension are used which would make sense in the theory of types with just three types. Though I may allude to this, it is not necessary to know anything about New Foundations to follow this talk.
The theory of types with three types is a 3-sorted theory with sorts called “type 0”, “type 1”, and “type 2”. Where $i$ is 0 or 1, $x^i \in y^{i+1}$ is a well-formed membership sentence. Where $i$ is 0,1,2, $x^i = y^i$ is a well-formed identity statement. We will not as a rule actually put type indices on variables; they will usually be deducible from context.
The axioms are Extensionality:

$$(\forall A.(\forall B.(\forall x.x \in A \leftrightarrow x \in B) \rightarrow A = B))$$

where the type of $A, B$ is one higher than the type of $x$, and Comprehension:

$$(\exists A.(\forall x.x \in A \leftrightarrow \phi)),$$

where $\phi$ is any formula in which $A$ does not occur free, the type of $A$ being one higher than the type of $x$. The object $A$ whose existence is asserted (unique by Extensionality) is called $\{x \mid \phi\}$. 
As noted above, we cannot define a relation or function as a set of ordered pairs in $TT_3$, because the ordered pair $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$ is only defined for $a$ and $b$ of type 0, and since it is a type 2 object it cannot be a member of any set in this theory.
We could attempt to define relations using sets of *unordered* pairs \(\{a, b\}\); of course, only symmetric relations could be defined in this way.

If \(\mathcal{R}\) is a symmetric relation (supposed defined by a formula) \((x \mathcal{R} y \leftrightarrow y \mathcal{R} x)\) then we can define \(\mathcal{R}\) (the set implementing \(\mathcal{R}\)) as

\[
\{\{a, b\} \mid a \mathcal{R} b\}.
\]

Of course, only relations on type 0 objects can be implemented in this way (and as a rule it is only structures on type 0 objects that we will attempt to implement).
We can implement any partial order $\leq$ as the collection of its (weak) segments: if $\mathcal{R}$ is a reflexive, antisymmetric, transitive relation, define the set $\mathcal{R}$ coding it as the collection

$$\{\{y \mid x \mathcal{R} y\} \mid x = x\}.$$ 

The weak segment is preferred because it is possible to distinguish between an ordering of a set with one element and an ordering of the empty set.
Quasi-Orders

More generally, any quasi-order (reflexive, transitive relation) can be implemented. If $\mathcal{R}$ is a reflexive, transitive relation defined by a formula, define the set $\mathcal{R}$ coding it as

$$\{\{y \mid x \mathcal{R} y\} \mid x = x\},$$

as above.

If $\mathcal{R}$ is a set, define $x \mathcal{R} y$ as $(\forall A \in \mathcal{R}. x \in A \rightarrow y \in A)$.

Quasi-orders include equivalence relations and partial orders, including linear orders and well-orderings.
Functions and Cardinality

In one special case the implementation of functions (and of the notion of cardinality) is quite easy. If \( A \) and \( B \) are disjoint sets, we can implement any function \( \mathcal{F} : A \to B \) (supposed defined by a formula \( \mathcal{F}(x, y) \) with appropriate properties, which we will equally often write \( y = \mathcal{F}(x) \)) using the set

\[
F = \{ \{ x, \mathcal{F}(x) \} \mid x \in A \}.
\]

The lack of directionality makes no difference, since we are not in doubt as to what is the domain and what is the range (if we were to interchange domain and range, exactly the same set would serve to represent \( \mathcal{F}^{-1} \)).
So we can define the assertion \( f : A \to B \) (\( f \) is a function from \( A \) to \( B \)), \( A \) and \( B \) being disjoint sets, as

\[
(\forall x \in A. (\exists y \in B. \{x, y\} \in f)) \land (\forall x \in A. (\forall y, z \in B. \{x, y\} \in f \land \{x, z\} \in f \rightarrow y = z)).
\]

We can define \( A \sim B \) as

\[
(\exists f. f : A \to B \land f : B \to A).
\]
Pabion used the preceding analysis of cardinality in his proof that $\text{NF}_3 + \text{Infinity}$ is equiconsistent with second order arithmetic, with the additional observation that for finite sets

$$A \sim B \iff A - B \sim B - A,$$

so the general case reduces to the disjoint case. (I am informed (later) that the observation is due to Maurice Boffa.)

Henrard showed (unpublished work) that cardinality can be defined for all sets, finite and infinite, disjoint and overlapping.
Our idea is to represent a function \( f \) using its set of forward orbits

\[
\{\{f^n(x) \mid n \geq 0\} \mid x \in \text{dom}(f)\}.
\]

This does not quite work, as we will see, but it does allow for a complete definition of cardinality and an “almost” complete definition of function.
Historical digression about Henrard’s approach

Henrard also used a representation of orbits in a bijection: he uses the idea of a “chain”, which is the set of unordered pairs \( \{x, f(x)\} \) in an orbit in the function. We look at how to express this concept without reference to functions: if \( A \) is a set of two-element sets such that no element of \( \bigcup A \) belongs to more than two elements of \( A \), then \( A \) is a union of chains. An element of \( \bigcup A \) which belongs to only one element of a union of chains \( A \) is called an endpoint of \( A \).
Historical digression about Henrard’s approach (cont.)

A closed chain is a nonempty union of chains with no endpoints, no proper subset of which is a nonempty union of chains with no endpoints. Any chain has the property that it has no proper subset which has no endpoints. A union of chains which has one endpoint and has no subset which is a closed chain is a chain. A union of chains which has two endpoints and which has no proper subset with no endpoints or one endpoint is a chain. These tools can be used to cover much the same ground as ours, however there is a disadvantage that there is no representation of the distinction between $f$ and $f^{-1}$ (for us, this distinction collapses only in (some) finite cycles in $f$).

The material about Henrard’s approach was added here after I had completed the work on the approach I present.
Our approach, continued

If $\mathcal{F}$ is any definable function (think of this as implemented by a formula $\mathcal{F}(x, y)$ with appropriate properties), define $O_x^\mathcal{F}$ as

$$\{y \mid (\forall A. x \in A \land (\forall z. z \in A \land z \in \text{dom}(\mathcal{F}) \rightarrow \mathcal{F}(z) \in A) \rightarrow y \in A)\}.$$ 

Define $F$ (the set coding $\mathcal{F}$) as

$$\{O_x^\mathcal{F} \mid (\exists y. y = \mathcal{F}(x)) \lor (\exists y. x = \mathcal{F}(y))\}.$$ 

Note that I do need orbits (taken to be singletons) for elements of the range of $\mathcal{F}$ which are not in the domain.
Orbits $O^F_x$ (elements of $F$) are of two kinds. There are finite sets (among which the sets of size 1 and 2 are special) and there are infinite sets. If $\{x\}$ is an orbit, then $\mathcal{F}(x) = x$. If $\{x, y\}$ is an orbit and $\{x\}$ is not, then $\mathcal{F}(x) = y$. From the other finite orbits, we cannot determine a function value.

If $O^F_x$ is not a finite set, then the distinguishing characteristic of $\mathcal{F}(x)$ is that $O^F_x - \{x\} = O^F_y$. To identify $O^F_x$ among the sets in $F$ (many of which contain $x$) observe that it is the intersection of all elements of $F$ that contain $x$. 
It is useful to pause and observe that the notion of finite set is definable in $TT_3$: the set $\text{Fin}$ can be defined as the set of all sets which contain $\emptyset$ and contain all sets $x \cup \{y\}$ whenever they contain $x$.

We have already noted that the notion of equinumerousness of finite sets is definable, so the cardinal of each type 1 finite set is already definable as a type 2 set.

Strictly speaking, one does not need to allude to the notion of finite set in the definitions which follow.
We define a first approximation to function application. Where $F$ is a set and $x$ is an element of $\bigcup F$, we define $F[x]$ as

$x$, in case $\{x\} \in F$

$y$, in case $\{x, y\} \in F$ and $\{x\} \not\in F$

For the next case, we need to define $O^F_x$ as $\cap \{A \in F \mid x \in A\}$:

the unique $y$ such that $O^F_y = O^F_x - \{x\}$, if this exists

else $x$, when none of the special conditions above hold.
If $\mathcal{F}$ is a definable function and $F$ is defined as above, $F[x] = \mathcal{F}(x)$ is true except in two special cases:

If $x$ is in the range of $\mathcal{F}$ but not in the domain of $\mathcal{F}$ then $F[x] = x$ will hold: knowledge of the intended domain and range of $\mathcal{F}$ makes this harmless.

More annoyingly, if $x$ is in a finite orbit in $\mathcal{F}$ with three or more elements, $F[x] = x$ rather than $\mathcal{F}(x)$. This is an essential obstruction to defining functions in three types which we cannot entirely overcome.
What is an obstruction to defining functions in general is not an obstruction to defining cardinality. If $\mathcal{F}$ is a definable bijection from $A$ to $B$, then $F$ with application defined as above will also be a bijection from $A$ to $B$. The fact that $F[x]$ is defined as $x$ on $B - A$ is harmless. Less obviously, the fact that $F[x]$ is defined as $x$ on finite orbits in $\mathcal{F}$ is also harmless: the reason that this is not a problem is that a finite orbit in $\mathcal{F}$ clearly must lie in $A \cap B$, and changing this to the identity, while it does change what bijection from $A$ to $B$ we consider, does not change the fact that it is a bijection from $A$ to $B$. 
So we can define $A \sim B$ in a quite standard way: 

$$(\exists F \mid (\forall x \in A. x \in \bigcup F \land F[x] \in B \land (\forall y \in A.F[x] = F[y] \rightarrow x = y) \land (\forall x \in B.(\exists! y \in A.F[y] = x))))).$$
It is important to consider whether we have the theory of cardinality that we expect. Is $\sim$, thus defined, an equivalence relation? Can we prove the Schröder-Bernstein theorem? The answer to both of these questions is yes, though the proofs are slightly different from the usual ones.
Equinumerousness is an Equivalence Relation

We prove that $\sim$ is an equivalence relation in more usual contexts by observing that the identity function on $A$ is a bijection witnessing $A \sim A$ (this works here), the fact that the inverse of a bijection from $A$ to $B$ is a bijection from $B$ to $A$ (this works here: if $F$ is a (set) bijection from $A$ to $B$, the relation $F[y] = x \land y \in A$ is bijective and (because $F$ is coded by a set) has no finite cycles of length greater than 2, so it is represented by a set).
The proof of transitivity uses the fact that the composition of a bijection from $B$ to $C$ with a bijection from $A$ to $B$ is a bijection from $A$ to $C$. This works, but not quite painlessly. Let $F$ be a bijection from $A$ to $B$ and $G$ be a bijection from $B$ to $C$, both coded by sets. Let $\mathcal{H}(x, y)$ be defined as $y = G[F[x]] \land x \in A$. This relation is bijective, and so the set $H$ coding it is a bijection witnessing $A \sim C$; but it is not necessarily the composition of $G$ and $F$ (it may be corrected to eliminate finite cycles).
We can define $|A| \leq |B|$ as “there is a subset $C$ of $B$ such that $A \sim C$”. An important result in the usual theory of cardinals is that $|A| \leq |B| \land |B| \leq |A| \rightarrow |A| = |B|$ (where $|A| = |B|$ is of course synonymous with $A \sim B$). This is the Schröder-Bernstein theorem.

The proof has the same flavor as the last clause of the previous proof: a bijective relation is defined in the manner of the usual proof, but the function obtained in the end is not necessarily the expected function.
Suppose that $f : A \to B' \subseteq B$ and $g : B \to A' \subseteq A$ are sets coding bijections. For any set $A$ and function $f$ whose domain includes $A$, define $f^n A$ as $\{f[x] \mid x \in A\}$. Define $P$ as the intersection of all sets which contain every element of $A - g^n B$ and contain $g[f[z]]$ whenever they contain $z$. Define $\mathcal{H}(x, y)$ as $(x \in P \land f[x] = y) \lor (x \in A - P \land g[y] = x)$. This is a bijective relation from $A$ to $B$, and the set $H$ coding $\mathcal{H}$ will be a bijection from $A$ to $B$ (but not necessarily the expected one).
If we have the Axiom of Infinity (which we can express in various forms: \( \text{Fin} \neq V \) works), we can show that the cardinals of finite sets satisfy the Peano axioms, and define addition and multiplication in sensible ways. We can show that for any finite sets \( A \) and \( B \), there is a finite set \( B' \) disjoint from \( A \) and equinumerous with \( B \), and \( |A| + |B| = |A \cup B'| \) (this can be a definition or a theorem if addition is defined in the usual inductive fashion). There is a more complicated way to characterize \( |A||B| \), supposing wlog that \( A \) and \( B \) are disjoint. A set \( C \) disjoint from \( A \) and \( B \) will have this cardinality if there is a set \( M \) each element of which is a triple consisting of one element of \( A \), one element of \( B \), and one element of \( C \), such that any two element set with one element of \( A \) and one element of \( B \) is a subset of exactly one element of \( M \) and any element of \( C \) belongs to exactly one element of \( M \).
Moreover, although natural numbers are type 2 objects we can nonetheless code any definable class of natural numbers as a set by considering the type 2 set of all type 1 sets which belong to some element of the class of natural numbers. This representation works because the natural numbers are disjoint from one another. \( TT_3 + \) Infinity interprets second order Peano arithmetic; in fact it is equiconsistent with second-order arithmetic (and so is \( NF_3 + \) Infinity, but this is beyond the scope of this talk).
Having completed the theory of cardinality, we ask whether the theory of functions, which is slightly defective, can be repaired? The answer is that it can be partially but not completely repaired without additional information on the structure of the universe.

Can other applications of the theory of functions be carried out? We will find that we can develop the complete theory of similarity of well-orderings (order types) (and more generally the theory of isomorphism types of linear orders).
Refining the Definition of Function

We show how to refine the definition of function so that it works essentially as often as possible. The difficulty is with finite cycles of length $> 2$. Suppose that $\mathcal{F}(x, y)$ is a functional formula (so $\mathcal{F}(x)$ is first-order definable) and $\mathcal{G}(x)$ is a formula which is true of exactly one member of each finite cycle of length $> 2$ in $\mathcal{F}$ (and only of elements of such finite cycles). A new class function $\mathcal{F}'(x, y)$ is definable as $(\neg \mathcal{G}(x) \land \mathcal{F}(x, y)) \lor (\mathcal{G}(x) \land x = y)$. The function $\mathcal{F}'$ contains no finite cycles, and we define a set $F'$ coding it exactly as above. We redefine $F$ (the set coding $\mathcal{F}$) as $F' \cup \{\{\mathcal{F}(x), \mathcal{F}(\mathcal{F}(x))\} \mid \mathcal{G}(x)\}$. 
The new elements of $F$ tell us where the function “reenters” a finite loop which has been cut in the transition from $F$ to $F'$. The new two element sets are identifiable, because they are the only two element sets $A$ in $F$ which are subsets of an element $B$ of $F$ such that $B$ contains as a subset a singleton element of $F$ disjoint from $A$. Thus $F'$ can be recovered from $F$. We define $F(x)$ as $F'[x]$ except when $\{x\} \in F$ and there is a unique $y$ such that $\{y, F'[y]\} \in A$, $\{y, F'[y]\}$ is disjoint from $\{x\}$, and there are elements of $A$ which contain all of $x, y, F'[y]$. In this case $F(x)$ is defined as $y$. 
The difficulty with this definition is that in the absence of a certain amount of Choice, there might be definable functional relations which did not admit a definable selection of one element from each of their finite cycles. In such a case, this definition might not work.

If one has Choice for collections of disjoint finite sets, this definition will always work. Note that this implies that we can always code functions if Infinity does not hold.

If one has a linear order on the universe (a condition which we can express with our ability to code partial orders as sets) then one will always be able to define functions and moreover, one has a much simpler way to do it: let $A$ and $B$ be two disjoint three element sets, let $\leq$ be the linear order, and define $\langle a, b \rangle$ as $\{a, b\} \Delta A$ if $a \leq b$ and $\{a, b\} \Delta B$ otherwise. It will then be possible to define functions on type 0 as sets of ordered pairs in the usual way.
Functions can always be defined on restricted domains which happen to support linear orders. For example, the statement that two well-orderings (or indeed any two linear orderings) are isomorphic can be stated in the usual way, because the linear order on the domain of the first well-ordering can be used to select one element from each cycle in a bijective relation witnessing the isomorphism. That isomorphism is an equivalence relation is provable in exactly the usual way, because compositions of functions with linearly ordered domains can be defined without the finite cycle corrections.

We got sets coding cardinals of type 0 sets, because their elements are type 1, but we do not get objects coding ordinal numbers or linear order types, because the orderings themselves (always on type 0) are represented by type 2 sets and so cannot belong to further sets.
There can be no reliable notion of function in $TT_3$

Consider a model of $TT_4$ (just add one more type) in which there is a function $f$ (represented by a set of Kuratowski pairs as usual) whose domain is an infinite union of cycles of length 3.

We use a permutation method to modify this model. Consider all elements $A$ of the model (at any positive type) which have the property that for some finite subset $B$ of the domain of $f$, $A$ is invariant under all permutations of type 0 which are obtained by iterating $f$ 0-2 times independently on each cycle in $f$ which is a subset of $V - B$. It is straightforward to show that these sets (the sets of finite support with respect to a certain group of permutations of type 0) make up a model of $TT_4$. This is essentially the Frankel-Mostowski method for
showing that Choice is independent of \( ZF \) with atoms.

Note that \( f \) has finite support (in fact, empty support) with respect to the group, so it still exists in the model. But the new model cannot contain any choice set for the finite orbits under \( f \), since all but finitely many of the orbits must contain 0 or 3 of the elements of any given set (because all but finitely many of the orbits are outside the support of the set).
Thus we cannot expect to code $f$ as above. But we can say more: we cannot code $f$ at all in the first three types. For any set $X$ of type 1, we have already seen that the elements of all but finitely many orbits have exactly the same relations to $X$; similarly, for any type 2 set $X$ and all but finitely many orbits $\{a, b, c\}$, and for any $Y$ of type 1 not containing any of $a, b, c$, either all of $\{a, b\} \cup Y, \{a, c\} \cup Y, \{b, c\} \cup Y$ are in $X$ or none of them are, and similarly for $\{a\} \cup Y, \{b\} \cup Y, \{c\} \cup Y$. In fact, the sets with finite support in our group in types 1-2 are the same as the sets with finite support in the larger group containing all permutations of each of the orbits independently. This means that there can be no 3-typed formula $H(x, y, f^*)$ equivalent to $y = f(x)$, since the pair $\langle x, y \rangle$ will have exactly the same relations to the parameter $f^*$ that the pair $\langle y, x \rangle$ does in all but finitely many cases where $y = f(x)$ is true. But the formula $y = f(x)$ (though it
involves 4 types) can be used as a “black box” in otherwise 3-typed formulas to define sets (since $f$ does have finite support and we still have a model of $TT_4$. If we had a function definition which worked for all 3-typed formulas, it would work to define $f$ in this case as well; so there can be no such definition.
Our conclusion is that $TT_3$ (and similarly $NF_3$) are powerful enough to specify arbitrarily complex mathematical structures without being powerful enough to actually prove the existence of (or explicitly construct) many structures. These theories allow the explicit construction of cardinal numbers and sets of cardinal numbers (though without commitment to any more than an initial segment of the finite cardinals, including all standard finite cardinals in the case of $NF_3$) but not much more. In this, these systems are more like second-order logic (also possessed of considerable expressive power but a very limited ability to construct anything without additional assumptions) than like more powerful set theories.