This homework will be due Tuesday, February 26. Please visit me during the intervening time, and remind me to open my grade book when you do, as I shall be officially noting your visits (no bonus for extra visits, though you are encouraged to visit as many times as needed).

You do not need to complete all the problems to get a good grade on this assignment. In particular, I think the last two may prove quite difficult for you (though I might be wrong!). The actual criteria will be determined by class performance.

1. You are given that $n > 0$ is a natural number and a, b are not natural numbers.

   Compute the Quine pairs $\langle x, y\rangle$ and $\langle y, x\rangle$ where $x = \{\{\emptyset, 3\}, \{2\}, \{0, b\}\}$ and $y = \{\{1, 2\}, \{n, a\}\}$

   Given that $\langle u, v\rangle = \{\{0, 2, 4\}, \{a, b, 2\}, \{0\}, \{1\}, \{a, n\}\}$, what are the sets u and v?

2. I give definitions of injective and surjective function from A to B (not identical to those in the book, though you are welcome to verify that they are equivalent).

   A function f is an injective function from A to B iff it is a function from A to B and for all $x, y \in A$, $f(x) = f(y) \rightarrow x = y$.

   A function f is a surjective function from A to B iff it is a function from A to B and for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

   Prove that if f is an injective function from A to B and g is an injective function from B to C, then $g \circ f$ is an injective function from A to C. ($g \circ f$ may be supposed defined by the equation $(g \circ f)(x) = g(f(x))$).

   Prove that if f is an surjective function from A to B and g is a surjective function from B to C, then $g \circ f$ is an surjective function from A to C.

   Use the definitions given in the problem and proof strategy as described in section 2.

   Comment: of course this shows compositions of bijections are bijections, which will be useful.

3. We define an *initial segment of the natural numbers* as a set S of natural numbers which has the property that for all natural numbers m, if $m + 1 \in S$ then $m \in S$.

Does an initial segment of the natural numbers need to contain all natural numbers? Explain why, or why not (with an example).

Prove that any nonempty initial segment of the natural numbers includes 0.

This is a proof in type theory (or second-order Peano arithmetic, with sets). How do we prove *anything* about natural numbers?

4. Prove as many of the following as you can in first-order Peano arithmetic, not necessarily in the given order. Your proofs should not mention sets or the type theory definitions of the natural numbers (this is all just arithmetic from the Peano axioms).

   Use proof strategy. You can be a little more freeform than in Homework 1, but take pains to make it clear what you are doing. You may use theorems already proved in the notes or already proved by you. You may *not* use anything else you think you know about arithmetic.

   Do prove at least one of them.

   The associative law of addition.

   The commutative law of multiplication.

   The associative law of multiplication.

   The distributive law of multiplication over addition.

5. Find sets A and B such that $A + 1 = B + 1$ but $A \neq B$. I found an example that isn't too hard to describe where $A + 1 = B + 1 = 3$ (or any natural number; nothing special about 3). This may turn out to be quite hard for you; I'm not certain. This shows that Axiom 4 is true of natural numbers but not of sets in general.

6. If I define a function I_n such that $I_n(f) = f^n$ (so for example $I_3(f)(x) = f^3(x) = f(f(f(x)))$, I invite you to consider the functions $(I_n)^m$. For example, compute $(I_2)^3(f)(x)$. Compute $(I_3)^2(f)(x)$. There is an equation $(I_m)^n = I^{F(m,n)}$, where F is a quite familiar operation on natural numbers, which you can write and might derive if you do enough experiments. There is a serious formal problem with this equation, though, in our type theory. What is the function $F(m, n)$? What is the formal problem?